M.Sc Part - II

Paper - I

ADVANCE MATHEMATICS ANALYSIS

Original Notes

Traders

GENTLEMAN

HOUSE OF QUALITY NOTES

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ADVANCE ANALYSIS

SECTION - I

SET THEORY

EQUIVALENT SETS:-

Two sets A and B are said to be equivalent if there exist bijective mapping from A to B i.e. there exist function $f: A \to B$, which is both 1-1 and onto. Than we write $A \sim B$.

EXAMPLE:- Let $f: N \to W$ be defined by
$$f(x) = x - 1.$$
then f is bijective function so
$$N \sim W.$$

INFINITE SET:-

A set is said to be infinite set if it is equivalent to its some proper subset.

e.g.

(i) Set of natural numbers N is infinite because $A = \{5, 6, \ldots\} \subset N$ and $f: N \to A$ defined by $f(n) = n + 4$ is bijective

$\Rightarrow N \sim A$

$\Rightarrow N$ is infinite

(ii) Set R of real numbers is infinite

Define $f:]-\pi/2, \pi/2[\to R$ by
$$f(x) = \tan x$$
than $f(x_1) = f(x_2)$
$$\Rightarrow \tan x_1 = \tan x_2$$

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is $1-1$

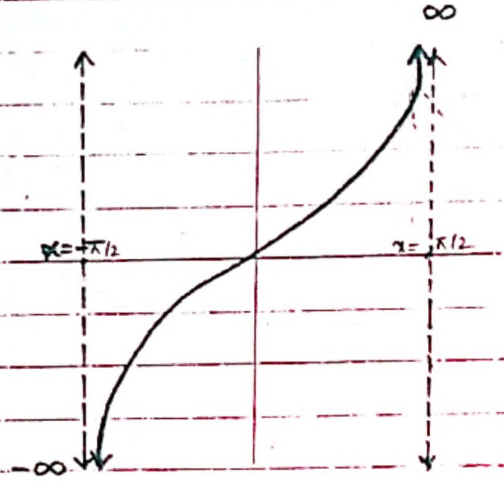
Obviously f is onto

$\Rightarrow f$ is bijective

$\Rightarrow]-\pi/2, \pi/2[\sim \mathbb{R}$

$\Rightarrow \mathbb{R}$ is infinite

also



from graph, f is bijective

$\Rightarrow \mathbb{R}$ is infinite

DENUMERABLE SETS:-

A set is said to be denumerable if it is equivalent to set of natural numbers.

e.g.

(i) Set of whole numbers (W) is denumerable.

(ii) Set of integers ($\mathbb{Z}$) is denumerable.

Pg # 22

QUESTION:- $\mathbb{Z}$ is denumerable.

Sol:-

For this we show $N \sim \mathbb{Z}$

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 0 & 1 & -1 & 2 & -2 & 3 & -3 & \ldots \end{array}$$

i.e. $f(x) = \begin{cases} \frac{x}{2} & , \text{if } x \text{ is even} \\ \frac{1-x}{2} & , \text{if } x \text{ is odd} \end{cases}$

Than obviously f is bijective

$\Rightarrow N \sim \mathbb{Z}$

$\Rightarrow N$ is denumerable.

QUESTION:- An infinite sequence with distinct elements is denumerable.

Pg # 25

SOLUTION:-

Let $\{a_n\}_{n=1}^{\infty}$ be an infinite sequence with distinct elements. Further let $A = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ be the set of the point of the sequence.

Define $f: N \to A$ by $f(i) = a_i$

than $f(i_1) = f(i_2)$

$\Rightarrow a_{i_1} = a_{i_2}$

$\Rightarrow i_1 = i_2$

$\Rightarrow f$ is 1-1

For every i, $a_i \in A$ we have $i \in N$

s.t $f(i) = a_i$

$\Rightarrow$ f is onto

$\Rightarrow$ f is bijective

$\Rightarrow$ A is denumerable.

Hence the result.

COUNTABLE SET:-

A set 'A' is said to be countable if it is either finite or denumerable.

NON DENUMERABLE SET:-

An infinite set A is said to be non denumerable if it is not equivalent to the set of natural numbers.

THEOREM:-

If A and B are two denumerable sets then $A \times B$ is also denumerable.

Pg # 26

PROOF:-

let $A = \{a_1, a_2, a_3, \ldots\}$ & $B = \{b_1, b_2, b_3, \ldots\}$

Now,

$$A \times B = \{(a_1, b_1), (a_1, b_2), (a_1, b_3), \ldots\ldots$$
$$(a_2, b_1), (a_2, b_2), (a_2, b_3), \ldots\ldots$$
$$(a_3, b_1), (a_3, b_2), (a_3, b_3), \ldots\ldots\}$$

The above arrow signs

$$A \times B = \{(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_1, b_3), (a_2, b_2), (a_3, b_1), \ldots\}$$

Then the elements of $A \times B$ can be regarded as elements of an infinite sequence with distinct elements.

Hence $A \times B$ is denumerable.

THEOREM:-

let $\{A_i\}$ be a family of pairwise disjoint denumerable sets then $\bigcup_i A_i$ is denumerable.

PROOF:-

Let $A_i = \{a_{i1}, a_{i2}, a_{i3}, \ldots\}$ Pg #27

Then $\bigcup_i A_i = \{a_{11}, a_{12}, a_{13}, \ldots$

$a_{21}, a_{22}, a_{23}, \ldots$

$a_{31}, a_{32}, a_{33}, \ldots \}$

Now,

As we know that if A and B are denumerable then $A \times B$ is denumerable.

So, As $\mathbb{N}$ is denumerable

$\Rightarrow \mathbb{N} \times \mathbb{N}$ is denumerable.

Now define

$f : \bigcup_i A_i \to \mathbb{N} \times \mathbb{N}$ by

$f(a_{ij}) = (i, j)$

Then,

$f(a_{i_1 j_1}) = f(a_{i_2 j_2})$

$(i_1, j_1) = (i_2, j_2)$

$\Rightarrow i_1 = i_2$ & $j_1 = j_2$

$\Rightarrow a_{i_1 j_1} = a_{i_2 j_2}$

$\Rightarrow f$ is 1 – 1

Since every $(i, j) \in \mathbb{N} \times \mathbb{N}$ is the image

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of some $a_{ij} \in \bigcup_i A_i$ i.e.
$f(a_{ij}) = (i, j)$
$\Rightarrow f$ is onto
$\Rightarrow f$ is bijective
$\Rightarrow \bigcup_i A_i \sim N \times N$
$\Rightarrow \bigcup_i A_i$ is denumerable.

THEOREM:-

If A is countable, B is denumerable, $A \cap B = \phi$ Then $A \cup B$ is denumerable.

PROOF:-

Here arises two cases for A.

Case I:-

If A is denumerable, then
$A = \{a_1, a_2, a_3, \ldots\}$
let $B = \{b_1, b_2, b_3, \ldots\}$
As $A \cap B = \phi$ So $A \cup B = \{a_1, a_2, \ldots; b_1, b_2, \ldots\}$

Define
$f : N \to A \cup B$ by

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \ldots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ a_1 & b_1 & a_2 & b_2 & a_3 & b_3 & \ldots \end{array}$$

Then
$N \sim A \cup B$
$\Rightarrow A \cup B$ is denumerable.

Case II :-

If A is finite.

Then let $A = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$
& $B = \{b_1, b_2, b_3, \ldots\ldots\}$

As $A \cap B = \phi$

So $A \cup B = \{a_1, a_2, a_3, \ldots\ldots, a_n, b_1, b_2, b_3, \ldots\ldots\}$

Define

$f : (A \cup B) \to N$ by

$f(a_i) = i$ & $f(b_i) = n+i$

Then obviously f is bijective

$$\Rightarrow \begin{array}{cccccccc} a_1 & a_2 & \cdots & a_n & b_1 & b_2 & \cdots \\ \updownarrow & \updownarrow & & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & \cdots & n & n+1 & n+2 & \cdots \end{array}$$

$\Rightarrow A \cup B \sim N$

$\Rightarrow A \cup B$ is denumerable.

REMARK:-

Countable union of countable sets is countable.

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THEOREM:- Every infinite set contains a subset which is denumerable.

PROOF :-

Let A be an infinite set. To prove A contains a set which is denumerable.

let $a_1 \in A$ Then $\{a_1\} \subseteq A$

Next let $a_2 \in A \setminus \{a_1\} \Rightarrow \{a_1, a_2\} \subseteq A$

let $a_3 \in A \setminus \{a_1, a_2\} \Rightarrow \{a_1, a_2, a_3\} \subseteq A$

Continuing in this way we have,

$B = \{a_1, a_2, a_3, ----, a_n, ------\} \subseteq A$

The existence of B is guaranteed by the fact that A is infinite & $a_1, a_2, a_3, -----, a_n, ----$ are distinct.

Then, B is denumerable

($\because$ An infinite sequence with distinct elements is denumerable)

THEOREM:-

Every subset of a denumerable sets is either finite or denumerable (i.e. countable).

PROOF:-

let $A = \{a_1, a_2, a_3, ...\}$ be a denumerable set and B be a subset of A

To prove B is either finite or denumerable.

If $B = \phi$ Then, B is finite.

If $B = A$ then, B is denumerable.

In both cases theorem is proved.

If $B \subset A$ and $B \neq \phi$ Then,
let $a_{i_1} \in B$, $a_{i_2} \in B \setminus \{a_{i_1}\}$,
$a_{i_3} \in B \setminus \{a_{i_1}, a_{i_2}\}$, ...

$\Rightarrow a_{i_1}, a_{i_2}, a_{i_3}, \ldots \in B$
If B is finite then, theorem is proved.

If B is infinite then,
$$B = \{a_{i_1}, a_{i_2}, a_{i_3}, \ldots\}$$
Then B is denumerable.
Hence theorem is proved.

QUESTION:- I.M.P

Set of Rational number is denumerable.

Solution:-

We know that $Q = Q^+ \cup \{0\} \cup Q^-$
Define
$$f : Q^+ \rightarrow N \times N \text{ by}$$
$$f(p/q) = (p, q)$$

$$\Rightarrow f(p_1/q_1) = f(p_2/q_2)$$
$$\Rightarrow (p_1, q_1) = (p_2, q_2)$$
$$\Rightarrow p_1 = p_2 \quad \& \quad q_1 = q_2$$
$$\Rightarrow \frac{p_1}{q_1} = \frac{p_2}{q_2}$$
$$\Rightarrow f \text{ is } 1-1$$

As for every $(p,q) \in N \times N$ we have $p/q \in Q^+$ s.t $f(p/q) = (p,q)$

$\Rightarrow f$ is onto

$\Rightarrow f$ is bijective

$\Rightarrow Q^+ \sim N \times N$

$\Rightarrow Q^{-1}$ is denumerable.

Now define $f : Q^+ \to Q^-$ by

$f(p/q) = -p/q$

Then $f(p_1/q_1) = f(p_2/q_2)$

$\Rightarrow -p_1/q_1 = -p_2/q_2$

$\Rightarrow p_1/q_1 = p_2/q_2$

$\Rightarrow f$ is 1 1

$\because$ for every $-p/q \in Q^-$, we have $p/q \in Q^+$ s.t $f(p/q) = -p/q$

$\Rightarrow f$ is onto

$\Rightarrow f$ is bijective.

$\Rightarrow Q^+ \sim Q^-$

$\Rightarrow Q^-$ is denumerable

As $\{0\}$ is countable, Q^+, Q^- are denumerable

Q^+, Q^- and $\{0\}$ are pairwise disjoint.

So $Q^+ \cup \{0\} \cup Q^-$ is denumerable

$\Rightarrow Q$ is denumerable.

REMARK:

$Q \times Q$ is denumerable.

QUESTION:- Show that the set of points in the plane with rational coordinate is denumerable.

Solution:-

Let S be the collection of such points. Then, $f : S \to \mathbb{Q} \times \mathbb{Q}$ defined by $f(p) = (r, s)$ where r and s are the coordinates of point 'p'.

Then $f(p_1) = f(p_2)$

$\Rightarrow (r_1, s_1) = (r_2, s_2)$

$\Rightarrow r_1 = r_2$ and $s_1 = s_2$

$\Rightarrow p_1 = p_2$

$\Rightarrow f$ is 1-1

$\because$ As for every $(r, s) \in \mathbb{Q} \times \mathbb{Q}$, we have $p \in S$ s.t $f(p) = (r, s)$

$\Rightarrow f$ is onto

$\Rightarrow f$ is bijective.

$\Rightarrow S \sim \mathbb{Q} \times \mathbb{Q}$

$\Rightarrow S$ is denumerable

$\Rightarrow$ Set of points in the plane with rational coordinate is denumerable.

QUESTION:-

Show that set P of all polynomials with integral coefficient is denumerable.

Solution:-

Consider $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_m x^m$
and let $|a_0| + |a_1| + |a_2| + \cdots + |a_m| = n$
and denote this polynomial with $P(n,m)$
i.e.

$$P(n,m) = a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m$$

with $|a_0| + |a_1| + |a_2| + \cdots + |a_m| = n$

Than

$$P = \bigcup_{(n,m) \in N \times N} P(n,m)$$

Now as for each pair (n,m), $P(n,m)$ are countable and countable union of countable sets is countable so, P is countable.
As P is not finite so, P is denumerable.

QUESTION:-

Show that set A of polynomial all algebraic numbers is denumerable.

Solution:-

(Solution of Polynomial equations are called algebraic numbers)

let $A_n = \{x : x \text{ is the solution of } P_n(x)=0\}$

Since $P_n(x)$ i.e. a polynomial of degree n equation can have at most n-solutions. So for each n, A_n is finite and hence is countable.

Now, $A = \bigcup_{n \in N} A_n$ Then, A is countable because countable union of countable sets is countable.

As A is not finite so, A is denumerable.

QUESTION:- Family of all pairwise disjoint intervals of real numbers is countable.

Solution:-

let $\{T_i : i \in I\}$ be a family of pairwise disjoint intervals of real numbers

To prove $\{T_i : i \in I\}$ is countable.

Now, As each interval T_i contains both rational and irrational numbers so, In particular each interval T_i contains a rational number q_i

Then, As for $i \neq j$, T_i and T_j are disjoint i.e. $T_i \cap T_j = \phi$

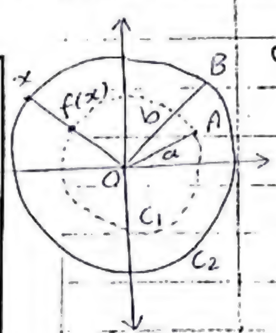
So $q_i \neq q_j$, where $q_i \in T_i$ and $q_j \in T_j$
Now define
$f : \{T_i : i \in I\} \longrightarrow \mathbb{Q}$ by
$f(T_i) = q_i$
Then f is 1 1. Now, It is also obvious that Rang $f \subseteq \mathbb{Q}$
As $\mathbb{Q}$ is countable and subset of a countable set is countable So Rang f is countable.

$\Rightarrow \{T_i : i \in I\} \sim$ Rang f
$\Rightarrow \{T_i \in i \in I\}$ is countable.

QUESTION:-

Consider
$C_1 = \{(x,y) : x^2+y^2 = a^2\}$
$C_2 = \{(x,y) : x^2+y^2 = b^2\}$ & $a<b$
Then show that
$C_1 \sim C_2$

$f : C_2 \to C_1$



$|OA| = a$
$|OB| = b$

Solution:
Define a function $f : C_2 \longrightarrow C_1$ by
" for every $x \in C_2$, $f(x)$ is the point of intersection of Circle C_1 and radius of Circle C_2 from centre of C_2 to point x on C_2."
Then for every point on C_1, we have radius from centre to that point & by producing this radius a point on C_2 which satisfies the definition of the function. Hence f is onto.

Further for $x_1 \neq x_2 \Rightarrow$ different radii from centre to x_1 & x_2 $\Rightarrow$ different point of intersections $\Rightarrow f(x_1) \neq f(x_2)$
$\Rightarrow f$ is 1-1 $\Rightarrow f$ is bijective.
$\Rightarrow C_2 \sim C_1 \Rightarrow C_1 \sim C_2$

CHARACTERISTIC FUNCTION:-

let $X \neq \phi$ and $A \subseteq X$ then, the function denoted and defined by

$$\chi_A : X \to \{0,1\}$$

$$\chi_A(x) = 1 \quad \text{if } x \in A$$
$$= 0 \quad \text{if } x \notin A$$

is called characteristic function on X mod A.

Set of all characteristic function on X is usually denoted by $C(X)$.

THEOREM:-

With usual meanings show that

$$2^X \sim C(X)$$

$C(X)$ = Collection of all characteristic functions defined on X.

2^X = Power set of X = collection of all subsets of X.

PROOF:-

Define $f: 2^X \to C(X)$ by

$$f(A) = \chi_A$$

$A \subseteq X$

Now,

Let $f(A_1) = f(A_2)$

$$\chi_{A_1} = \chi_{A_2}$$

$\chi_{A_1}(x) = \chi_{A_2}(x) \;, \forall x \in X$

let $x \in A_1 \Rightarrow \chi_{A_1}(x) = 1$

$\Rightarrow \chi_{A_2}(x) = 1$

$\Rightarrow x \in A_2$

$\Rightarrow A_1 \subseteq A_2$ — (i)

Now let

$x \in A_2 \Rightarrow \chi_{A_2}(x) = 1$

$\Rightarrow \chi_{A_1}(x) = 1$

$\Rightarrow x \in A_1$

$\Rightarrow A_2 \subseteq A_1$ — (ii)

from (i) and (ii)

$A_1 = A_2$.

Hence f is 1 – 1

Now, As for every $\chi_A \in C(X)$, we have $A \in 2^X$ s.t $f(A) = \chi_A$

$\Rightarrow f$ is onto

$\Rightarrow f$ is bijective

$\Rightarrow 2^X \sim C(X)$

THEOREM:- 2013, 2014, 2019

Prove that $[0,1]$ is non denumerable.

PROOF:- Suppose on the contrary that

$[0,1]$ is denumerable

Then $[0,1] = \{x_1, x_2, x_3, \ldots\}$

Now, As for every $x \in [0,1]$ we can write $x = 0.a_1a_2a_3\ldots\ldots$

where each $a_i = \{0,1,2,\ldots,9\}$

So then let

$$x_1 = 0.a_{11}a_{12}a_{13}a_{14}a_{15}\ldots\ldots$$
$$x_2 = 0.a_{21}a_{22}a_{23}a_{24}a_{25}\ldots\ldots$$
$$x_3 = 0.a_{31}a_{32}a_{33}a_{34}a_{35}\ldots\ldots$$
$$\vdots$$

where for all i, j, $a_{ij} \in \{0,1,2,\ldots,9\}$

Now, consider an element

$$y = 0.b_1b_2b_3\ldots\ldots$$

s.t $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $b_3 \neq a_{33}\ldots\ldots$

and $b_1, b_2, b_3, \ldots\ldots \in \{0,1,2,\ldots,9\}$

As $y = 0.b_1b_2b_3\ldots\ldots$ and each $b_i \in \{0,1,2,\ldots,9\}$ so $y \in [0,1]$

As, $y = 0.b_1b_2b_3\ldots\ldots$

and $b_1 \neq a_{11} \Rightarrow y \neq x_1$

$b_2 \neq a_{22} \Rightarrow y \neq x_2$

& $b_3 \neq a_{33} \Rightarrow y \neq x_3$

$\vdots$

$$\Rightarrow y \notin \{x_1, x_2, x_3, \ldots\} = [0,1]$$
$$\Rightarrow y \notin [0,1]$$

which is a contradiction so our supposi-

-tion is wrong. Hence $[0,1]$ is non denumerable.

COROLLARY:-

Prove That $]0,1[$ is non denumerable.

PROOF:-

1st METHOD:-

To prove This we show that

$$]0,1[\sim[0,1]$$

we can write $[0,1]$ as follows:

$$[0,1]=\{0,1,1/2,1/3,\ldots\}\cup A$$

where $A=[0,1]\setminus\{0,1,1/2,1/3,\ldots\}$

Also we can write $]0,1[$ as follows

$$]0,1[=\{1/2,1/3,\ldots\}\cup A$$

where $A=[0,1]\setminus\{0,1,1/2,1/3,\ldots\}$

Now define

$f: [0,1]\to]0,1[$ by

$$f(x)=\begin{cases}1/2 & \text{if } x=0\\ \dfrac{1}{n+2} & \text{if } x=\dfrac{1}{n}\\ x & \text{if } x\in A\end{cases}$$

Obviously f is both one-one and onto.

$\Rightarrow f$ is bijective
$\Rightarrow \,]0,1[\sim [0,1]$
$\Rightarrow \,]0,1[$ is non denumerable.

2^{nd} METHOD:

Suppose on the contrary that $]0,1[$ is denumerable.
Then $[0,1] = \{0,1\} \cup \,]0,1[$ is denumerable
$\because$ A is countable, B is denumerable and $A \cap B = \phi$ then $A \cup B$ is denumerable
But $[0,1]$ is non denumerable
Gives contradiction, As $[0,1]$ is non denumerable So our supposition is wrong and hence $]0,1[$ is non denumerable.

COROLLARY:-

Prove that $]0,1]$ and $[0,1[$ are non denumerable.

Proof:-

To prove $]0,1]$ is non denumerable, first we prove that $[0,1]$ is non denumerable.

Suppose on the contrary that $[0,1]$ is denumerable. Then
$$[0,1] = \{x_1, x_2, x_3, \ldots\}$$
Now as for every $x \in [0,1]$ we can write $x = 0.a_1a_2a_3\ldots\ldots$

where each $a_i = \{0,1,2,\dots,9\}$
So then set

$x_1 = 0.a_{11}a_{12}a_{13}\dots$
$x_2 = 0.a_{21}a_{22}a_{23}\dots$
$x_3 = 0.a_{31}a_{32}a_{33}\dots$
$\vdots$

where for all i,j, $a_{ij} \in \{0,1,2,\dots,9\}$
Now consider an element
$y = 0.b_1b_2b_3\dots$

s.t $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, $b_3 \neq a_{33}\dots$
and $b_1, b_2, b_3, b_4, \dots \in \{0,1,2,3,\dots,9\}$

As $y = 0.b_1b_2b_3\dots$ and each b_i,
$b_i \in \{0,1,2,\dots,9\}$
So $y \in [0,1]$

As $y = 0.b_1b_2b_3\dots$
and $b_1 \neq a_{11} \Rightarrow y \neq x_1$
$b_2 \neq a_{22} \Rightarrow y \neq x_2$
$\vdots$

$\Rightarrow y \notin \{x_1, x_2, x_3, \dots\} = [0,1]$
$\Rightarrow y \notin [0,1]$
which is a contradiction, so our supposition is wrong and hence $[0,1]$ is non denumerable.

Now we will prove that $]0,1]$ is non denumerable.

As Suppose on the contrary $]0,1]$ is denumerable. Then,

$[0,1] =]0,1] \cup \{0\}$ is denumerable

$\because$ if A is denumerable and B is finite and $A \cap B = \phi$ Then $A \cup B$ is denumerable.

But $[0,1]$ is non denumerable, gives contradiction so, our supposition is wrong and hence $]0,1]$ is non denumerable.

(ii) $[0,1[$ is non denumerable.

Suppose on the contrary that $[0,1[$ is ~~non~~ denumerable

Then

$[0,1] = \{1\} \cup [0,1[$ is denumerable

$\because$ If A is finite and B is denumerable and $A \cap B = \phi$ Then $A \cup B$ is denumerable.

But $[0,1]$ is non denumerable.

Gives contradiction So our supposition is wrong and hence $[0,1[$ is non denumerable.

THEOREM:-

Prove that $[a,b]$ is non denumerable

PROOF:-

First we prove that $[0,1]$ is non denumerable.

Suppose on the contrary that $[0,1]$ is denumerable. Then,

$$[0,1] = \{x_1, x_2, x_3, \ldots\}$$

Now, as for every $x \in [0,1]$ we can write

$$x = 0.a_1a_2a_3\ldots$$

where each $a_i \in \{0,1,2,\ldots,9\}$

So then,

let $x_1 = 0.a_{11}a_{12}a_{13}\ldots$

$x_2 = 0.a_{21}a_{22}a_{23}\ldots$

$x_3 = 0.a_{31}a_{32}a_{33}\ldots$

$\vdots$

where for all i,j, $a_{ij} \in \{0,1,\ldots,9\}$

Now consider an element

$$y = 0.b_1b_2b_3\ldots$$

s.t $b_1 \neq a_{11}$, $b_2 \neq a_{22}$, ……

and $b_1, b_2, b_3, \ldots \in \{0,1,2,3,\ldots,9\}$

As $y = 0.b_1b_2b_3\ldots$ and each $b_i \in \{0,1,2,3,\ldots,9\}$

$\Rightarrow y \in [0,1]$

As $y = 0.b_1b_2b_3\ldots\ldots$
and $b_1 \neq a_{11} \Rightarrow y \neq x_1$
$b_2 \neq a_{22} \Rightarrow y \neq x_2$
$b_3 \neq a_{33} \Rightarrow y \neq x_3$
$\vdots$

$\Rightarrow y \notin \{x_1, x_2, x_3, \ldots\} = [0,1]$
$\Rightarrow y \notin [0,1]$.

which is a contradiction, So our supposition is wrong and hence $[0,1]$ is non denumerable.

Now Define $f : [0,1] \to [a,b]$
by $f(x) = a + (b-a)x$.
Then,
$f(x_1) = f(x_2)$
$\Rightarrow a + (b-a)x_1 = a + (b-a)x_2$
$\Rightarrow x_1 = x_2$
$\Rightarrow f$ is 1-1.

Now as for every $a + (b-a)x \in [a,b]$ we have $x \in [0,1]$ s.t. $f(x) = a + (b-a)x$
$\Rightarrow f$ is onto
$\Rightarrow f$ is bijective
$\Rightarrow [0,1] \sim [a,b]$
$\Rightarrow [a,b]$ is non denumerable.

COROLLARY:-

Prove that $]a,b[$, $]a,b]$, $[a,b[$ are all non denumerable.

PROOF:-

(i) $]a,b[$ is non denumerable.

Define $f:]0,1[\to]a,b[$ by

$$f(x) = a+(b-a)x.$$

Then

$$f(x_1) = f(x_2)$$
$$\Rightarrow a+(b-a)x_1 = a+(b-a)x_2$$
$$\Rightarrow x_1 = x_2$$
$$\Rightarrow f \text{ is } 1\text{-}1.$$

Now as for every $a+(b-a)x \in]a,b[$ we have $x \in]0,1[$ s.t. $f(x) = a+(b-a)x$

$\Rightarrow f$ is onto

$\Rightarrow f$ is bijective

$\Rightarrow]0,1[\sim]a,b[$

$\Rightarrow]a,b[$ is non denumerable.

(ii) $]a,b]$ is non denumerable.

Define $f:]0,1] \to]a,b]$ by

$$f(x) = a+(b-a)x$$

then $f(x_1) = f(x_2)$

$$\Rightarrow a+(b-a)x_1 = a+(b-a)x_2$$
$$\Rightarrow x_1 = x_2$$
$$\Rightarrow f \text{ is } 1\text{-}1$$

Now as for every $a+(b-a)x \in]a,b]$ we have $x \in]0,1]$ s.t $f(x) = a+(b-a)x$

$\Rightarrow$ f is onto

$\Rightarrow$ f is bijective

$\Rightarrow$ $]0,1] \sim]a,b]$

$\Rightarrow$ $]a,b]$ is non denumerable.

(iii) $[a,b[$ is non denumerable.

Defin $f : [0,1[\longrightarrow [a,b[$ by

$f(x) = a+(b-a)x.$

Then $f(x_1) = f(x_2)$

$\Rightarrow$ $a+(b-a)x_1 = a+(b-a)x_2$

$\Rightarrow$ $x_1 = x_2$

$\Rightarrow$ f is 1-1.

Now as for every $a+(b-a)x \in [a,b[$ we have $x \in [0,1[$, s.t $f(x) = a+(b-a)x$

$\Rightarrow$ f is onto

$\Rightarrow$ f is bijective.

$\Rightarrow$ $[0,1[\sim [a,b[$

$\Rightarrow$ $[a,b[$ is non denumerable.

THEOREM:- Prove that set of real number $\mathbb{R}$ is non denumerable.

PROOF:-

As $]-\pi/2, \pi/2[$ is non denumerable and $]-\pi/2, \pi/2[\sim \mathbb{R}$

$\Rightarrow \mathbb{R}$ is non denumerable.

THEOREM:- Prove that set Q' of irrational number is non denumerable.

PROOF:-

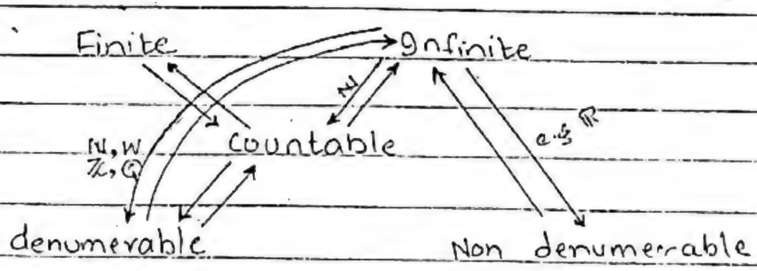
Suppose on the contrary that Q' is denumerable. Then as union of two denumerable disjoint sets is denumerable and Q is denumerable

So $Q \cup Q' = \mathbb{R}$ is denumerable which is a contradiction. ($\because \mathbb{R}$ is non denumerable) So our supposition is wrong and hence Q' is non denumerable.

- Sets
  - finite
    - countable
  - Infinite
    - denumerable
      - countable
    - non denumerable
      - uncountable

- Sets
  - countable
    - finite
    - Infinite and denumerable
  - uncountable
    - infinite and non denumerable

- Sets
  - denumerable
    - infinite and countable
  - non denumerable
    - infinite but uncountable



CARDINAL NUMBERS

In Mathmatics cardinal numbers or simply cardinals are generalized numbers used to measure the cardinality (size) of sets. For finite sets the cardinality is given by a natural number which is simply the number of elements in the set. There are also Transfinite cardinal numbers that describe the sizes of infinite sets.

Cardinality is defined in terms of bijective function. Two sets have the same cardinal numbers if and only if there is a bijection between them. In the case of finite sets, this agrees without any confusion because if the two sets have same cardinal numbers Than it means they have the same number of element and in this case one can easily find a bijection between the two sets. In case of infinite sets the behaviour is more complex.

A fundamental theorem by Georg cantor shows that it is possible for infinite sets to have different cardinalities and in particular the set of real numbers and the set of natural numbers do not have the same cardinal number, This is because they are not equivalent to each other. Further complication also

arises that in case of infinite set it is possible that a set and its proper subset both have same cardinality, which is not possible for finite sets.

Alongwith being a part of set theory, cardinality is also used in Combinatorics, Abstract algebra and Mathimatical Analysis.

Although there are many infinite cardinals but here we will deal within and upto the cardinality of $\mathbb{R}$ and its subsets.

Hence:

Cardinality of a set A means.

(i) Numbers of element in the set if A is finite set.

(ii) Cardinality of set A means, cardinality of other set B, if A is infinite or finite but $A \sim B$.

Here we discuss the cardinality of infinite sets by dividing them in two classes.

(i) Denumerable sets

(ii) Non denumerable sets.

Cardinality of any denumerable set in general & of $\mathbb{N}$ in particular is denoted by a or $\aleph_0$.

Similarly cardinality of any non denumerable set (subset of $\mathbb{R}$) in general and of $[0,1]$ in particular is denoted by c

$$\Rightarrow \#(\mathbb{N}) = a \quad \& \quad \#([0,1]) = c$$

ARITHMATIC OF CARDINALS:-

Let $\alpha = \#(A)$ & $\beta = \#(B)$

Then,

$\alpha + \beta = \#(A \cup B)$, provided $A \cap B = \phi$.

& $\alpha \cdot \beta = \#(A \times B)$

THEOREM:- For any cardinals α, β, γ

(i) $\alpha + \beta = \beta + \alpha$

(ii) $\alpha\beta = \beta\alpha$

(iii) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

(iv) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$

(v) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$

PROOF

(i) $\alpha + \beta = \beta + \alpha$

let A and B be any two sets s.t

$\#(A) = \alpha$, & $\#(B) = \beta$

& $A \cap B = \phi$

Then $\alpha + \beta = \#(A \cup B)$

$= \#(B \cup A)$ $\because A \cup B = B \cup A$

$= \beta + \alpha$

(ii) $\alpha\beta = \beta\alpha$

$\alpha\beta = \#(A \times B)$

& $\beta\alpha = \#(B \times A)$

Define $f: A \times B \to B \times A$ by

$f(a, b) = (b, a)$

The $f(a_1, b_1) = f(a_2, b_2)$

$\Rightarrow (b_1, a_1) = (b_2, a_2)$

$\Rightarrow b_1 = b_2$ & $a_1 = a_2$

$\Rightarrow (a_1, b_1) = (a_2, b_2)$

$\Rightarrow f$ is 1 – 1

Also for any $(b, a) \in B \times A$, we have $(a, b) \in A \times B$ s.t $f(a, b) = (b, a)$

$\Rightarrow f$ is onto

$\Rightarrow f$ is bijective

$\Rightarrow A \times B \sim B \times A$

$\Rightarrow \#(A \times B) = \#(B \times A)$

$\Rightarrow \alpha\beta = \beta\alpha$

(iii) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$

let A, B, C be the pair wise disjoint sets. s.t

$\alpha = \#(A)$, $\beta = \#(B)$ & $\gamma = \#(C)$

Now $\alpha + (\beta + \gamma) = \#[A \cup (B \cup C)]$

$= \# [(A\cup B)\cup C]$ $\because A\cup(B\cup C) = (A\cup B)\cup C$

$= (\alpha+\beta)+\gamma$

(iv) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$

Let $\#(A) = \alpha$, $\#(B) = \beta$, $\#(C) = \gamma$

Now, $\#[A\times(B\times C)] = \alpha(\beta\gamma)$

and $\#[(A\times B)\times C] = (\alpha\beta)\gamma$

Define function $f: A\times(B\times C) \to (A\times B)\times C$
by $f(a,(b,c)) = ((a,b),c)$

Then $f(a_1,(b_1,c_1)) = f(a_2,(b_2,c_2))$
$\Rightarrow ((a_1,b_1),c_1) = ((a_2,b_2),c_2)$

$\Rightarrow (a_1,b_1) = (a_2,b_2)$ & $c_1 = c_2$
$\Rightarrow a_1 = a_2$, $b_1 = b_2$ & $c_1 = c_2$

$\Rightarrow a_1 = a_2$, $(b_1,c_1) = (b_2,c_2)$
$\Rightarrow (a_1,(b_1,c_1)) = (a_2,(b_2,c_2))$
$\Rightarrow f$ is 1-1

Also for every $((a,b),c) \in (A\times B)\times C$, we have $(a,(b,c)) \in A\times(B\times C)$ s.t
$f(a,(b,c)) = ((a,b),c)$
$\Rightarrow f$ is onto
$\Rightarrow f$ is bijective
$\Rightarrow A\times(B\times C) \sim (A\times B)\times C$

$\Rightarrow \#(A\times(B\times C)) = \#((A\times B)\times C)$

$\Rightarrow \alpha(\beta\gamma) = (\alpha\beta)\gamma$

(v) $\alpha(\beta+\gamma) = \alpha\beta+\alpha\gamma$

Let A, B, C be the three sets s.t
$B\cap C = \phi$
and $\alpha = \#(A)$, $\beta = \#(B)$, $\gamma = \#(C)$
Note that as $B\cap C = \phi$, so
$(A\times B)\cap(A\times C) = \phi$

$$\alpha(\beta+\gamma) = \#[A\times(B\cup C)]$$
$$= \#[(A\times B)\cup(A\times C)]$$
$$= \alpha\beta+\alpha\gamma$$

INEQUALITIES IN CARDINAL NUMBERS :-

Let A and B be two sets with
$\#(A) = \alpha$ and $\#(B) = \beta$
Then
$\alpha < \beta$ (read as α is less then β),
if every 1-1 function $f: A \to B$
is not onto.

OR

If A is equivalent to some subset of B and A is not equivalent to B. Then we write $A \prec B$, and read it as A strictly preceeds B.

Further, $\alpha \le \beta$ if A is equivalent to some subset of B. Then we write $A \precsim B$ and read as A preceeds B.

QUESTION:-

With usual meanings show that $ac = c$

Sol:-

Let $a = \#(\mathbb{Z})$ and $c = \#([0,1]) = \#(A)$
where $A = [0,1]$

Define $f : \mathbb{Z} \times A \to \mathbb{R}$ by
$$f(n, b) = n + b$$

Then for every real number x, x can be divided into two parts. 1st one is integeral part and 2nd one is positive decimal part 'b' and $x = n + b$, then $(n, a) \in \mathbb{Z} \times A$
s.t. $f(n, b) = n + b = x$
$\Rightarrow f$ is onto

Now let $f(n_1, b_1) = f(n_2, b_2)$
$$\Rightarrow n_1 + b_1 = n_2 + b_2$$
$$\Rightarrow n_1 = n_2 \text{ and } b_1 = b_2$$
$$(\because a_1, a_2 \geq 0)$$

$\Rightarrow (n_1, b_1) = (n_2, b_2)$

$\Rightarrow f$ is 1-1

$\Rightarrow \mathbb{Z} \times A \sim \mathbb{R}$

$\Rightarrow \#(\mathbb{Z} \times A) = \#(\mathbb{R})$

$\Rightarrow ac = c$

REMARK:-

Show by an example that

(i) Cancellation laws do not hold for cardinal addition.

(ii) Cancellation laws do not hold for cardinal multiplication.

SOLUTION:-

(i)

Exp:-

Let $\#(\mathbb{N}) = a$ and $A = \phi$

$B = \{x\}$ and $x \notin \mathbb{N}$

Then

$\mathbb{N} \cup A = \mathbb{N}$ and $\mathbb{N} \cup B = \{x, 1, 2, 3, \ldots\}$

Then $\#(\mathbb{N} \cup A) = \#(\mathbb{N})$

$\#(\mathbb{N}) + \#(A) = \#(\mathbb{N})$

$a + 0 = a \longrightarrow \star$

Now define

$f : \mathbb{N} \to \mathbb{N} \cup B$ by

$f(1) = x,\ f(2) = 1,\ f(3) = 2, \ldots\ldots$

Then f is bijective.

$\Rightarrow \mathbb{N} \sim \mathbb{N} \cup B \Rightarrow \#(\mathbb{N}) = \#(\mathbb{N} \cup B)$

$\Rightarrow \#(\mathbb{N}) = \#(\mathbb{N}) + \#(B)$

$\Rightarrow a = a + 1 \longrightarrow **$

from $*$ and $**$

$a + 0 = a + 1$

$\Rightarrow 0 = 1$

(ii)

Exp.

Now $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

$\Rightarrow \#(\mathbb{N} \times \mathbb{N}) = \#(\mathbb{N})$

$\Rightarrow aa = a \longrightarrow *$

Also $\mathbb{N} \times \{1\} \sim \mathbb{N}$ by $f(n, 1) = n$

$\Rightarrow \#(\mathbb{N} \times \{1\}) = \#(\mathbb{N})$

$\Rightarrow a.1 = a \longrightarrow **$

from $*$ and $**$

$a.a = a.1$

$\Rightarrow a = 1$

THEOREM:-

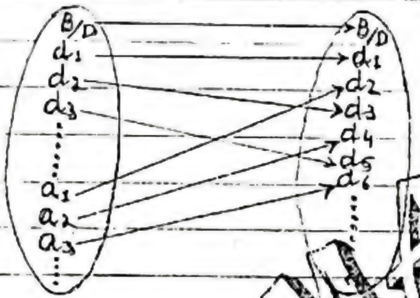
let β be any infinite cardinal numbers, then $a + \beta = \beta$

PROOF:-

let B be any infinite set with $\#(B) = \beta$ and $A = \{a_1, a_2, a_3, \ldots\}$ is a denumerable set, and $\#(A) = a$ and $A \cap B = \phi$

Now, as B is an infinite set and we know that any infinite set contains a denumerable set say

$D = \{d_1, d_2, d_3, \ldots\}$ be a denumerable subset of B. Then define $f: A \cup B \to B$ by



i.e. $f(x) = x$ if $x \in B/D$
$= d_{2n-1}$ if $x = d_n$
$= d_{2n}$ if $x = a_n$

$f(x) = x$ if $x \in B/D$
$= d_{2n-1}$ if $x = d_n$
$= d_{2n}$ if $x = a_n$

Then f is bijective.
and $\Rightarrow A \cup B \sim B$
$\Rightarrow \#(A \cup B) = \#(B)$
$\Rightarrow \alpha + \beta = \beta$

THEOREM:- Prove that $c^2 = c$

PROOF:

Let $A = [0, 1]$ Then $c^2 = c \cdot c = \#(A \times A)$

Now, Let $(x, y) \in A \times A \Rightarrow x, y \in A$

$x = 0.x_1x_2x_3\ldots$ and $y = 0.y_1y_2y_3\ldots$

all $x_i, y_i \in \{0, 1, 2, 3, \ldots, 9\}$.

Define $f: A \times A \to A$ by

$f(x, y) = 0.x_1y_1x_2y_2\ldots$

Then $f(x, y) = f(x', y')$

$x = 0.x_1x_2x_3\ldots\ldots$

$y = 0.y_1y_2y_3\ldots\ldots$

& $x' = 0.x_1'x_2'x_3'\ldots\ldots$ and $y' = 0.y_1'y_2'y_3'\ldots\ldots$

then $f(x,y) = f(x',y')$

$\Rightarrow 0.x_1y_1x_2y_2x_3y_3\ldots\ldots = 0.x_1'y_1'x_2'y_2'\ldots\ldots$

$\Rightarrow x_1 = x_1', y_1 = y_1', y_2 = y_2', \ldots\ldots$

$\Rightarrow 0.x_1x_2x_3,\ldots\ldots = 0.x_1'x_2'x_3',\ldots\ldots$

and $0.y_1y_2y_3\ldots\ldots = 0.y_1'y_2'y_3'\ldots\ldots$

$\Rightarrow x = x'$ and $y = y'$

$\Rightarrow (x,y) = (x',y')$

$\Rightarrow f$ is 1 1

$\cancel{\Rightarrow \#(A) \leq \#(B)}$

$\Rightarrow \#(A\times A) \leq \#(A)$

$\Rightarrow c^2 \leq c$

Now define

$f: A \rightarrow A\times A$ by

$f(x) = (0,x)$

The f is 1 1

$\Rightarrow \#(A) \leq \#(A\times A)$

$\Rightarrow c \leq c^2$

$\Rightarrow c^2 \leq c \leq c^2$

$\Rightarrow c^2 = c$

QUESTION:- Show that $2^a = c$.

Sol:-

Define $f: \mathbb{R} \to 2^{\mathbb{Q}}$ by

$$f(x) = \{q : q \in \mathbb{Q} \text{ and } q < x\}$$

Then if $x, y \in \mathbb{R}$ s.t $x \neq y$

then say $x < y$

Then by rational density theorem there exist some rational number r, s.t $x < r < y$

As $r < y$ so $r \in f(y)$

and $r > x$ i.e $r \nless x \Rightarrow r \notin f(x)$

$\Rightarrow f(x) \neq f(y)$

$\Rightarrow f$ is 1 1.

$\Rightarrow \mathbb{R} \precsim 2^{\mathbb{Q}}$

$\Rightarrow \#(\mathbb{R}) \leq \#(2^{\mathbb{Q}})$

$\Rightarrow c \leq 2^a \longrightarrow (1)$

Now, Consider $C(\mathbb{N})$, set of characteristic function defined on $\mathbb{N}$ and define, $F: C(\mathbb{N}) \to [0,1]$ by

$$F(f) = 0.f(1)f(2)f(3)\ldots\ldots$$

let $f, g \in C(\mathbb{N})$ s.t $f \neq g$

Then There exist at least one $n \in \mathbb{N}$ s.t $f(n) \neq g(n)$

$\Rightarrow 0.f(1)f(2)f(3)\ldots\ldots \neq 0.g(1)g(2)g(3)\ldots\ldots$

$\Rightarrow F(f) \neq F(g)$

$\Rightarrow F$ is 1 1

$\Rightarrow C(N) \not\sim [0,1]$

but $C(N) \sim 2^N$

$\Rightarrow 2^N \not\sim [0,1]$

$\Rightarrow \#(2^N) \leq \#[0,1]$

$\Rightarrow 2^a \leq c \longrightarrow$ (ii)

From (i) and (ii)

$2^a = c$

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CONTINUUM HYPOTHESIS :-

There does not exist any cardinal β s.t $a < \beta < c$.

SCHRODER BERNSTEIN THEOREM :-

If $X_1 \subseteq Y \subset X$ and $X \sim X_1$ then $X \sim Y$

PROOF :-

Since $X \sim X_1$ so then there exist $f: X \to X_1$ which is both 1-1 and onto.

Therefore as $Y \subseteq X$ so, restriction of f to Y is also 1-1, here we denote restriction by f also. i.e

$f: Y \to X_1$ is 1-1. Then there exist some subset Y_1 of X_1 s.t $f: Y \to Y_1$ is bijective i.e $Y \sim Y_1$

Note that $Y_1 \subset X_1 \subseteq Y \subseteq X$. Further as $X_1 \subseteq Y$ and (Y is equivalent to Y_1) i.e $Y \sim Y_1$, then by above

argument X_1 is equivalent to some subset of Y_1 say X_2 i.e $X_1 \sim X_2$
and $X_2 \subset Y_1 \subseteq X_1 \subseteq Y \subseteq X$
Further, as $Y_1 \subseteq X_1$ and $X_1 \sim X_2$ so again by above argument there exist some subset Y_2 of X_2 s.t $Y_1 \sim Y_2$
and $Y_2 \subset X_2 \subset Y_1 \subset X_1 \subset Y \subset X$
Continuing this process we obtain the sequences of equivalent sets
$X_1, X_2, X_3, \ldots\ldots$ and $Y_1, Y_2, Y_3, \ldots\ldots$ &

$$\ldots\ldots\ldots \subseteq Y_2 \subseteq X_2 \subset Y_1 \subseteq X_1 \subseteq Y \subseteq X$$

let $B = X \cap Y \cap X_1 \cap Y_1 \cap X_2 \cap Y_2 \ldots\ldots\ldots$

Then, $X = (X-Y) \cup (Y-X_1) \cup (X_1 - Y_1) \cup (Y_1 - X_2) \cup \ldots \ldots\ldots \cup B$

and $Y = (Y-X_1) \cup (X_1-Y_1) \cup (Y_1-X_2) \cup \ldots\ldots \cup B.$

Also Note that $(X-Y) \overset{f}{\sim} (X_1-Y_1) \overset{f}{\sim} (X_2-Y_2) \overset{f}{\sim} \ldots\ldots\ldots$

Specifically $f: X_n - Y_n \to X_{n+1} - Y_{n+1}$ is bijective

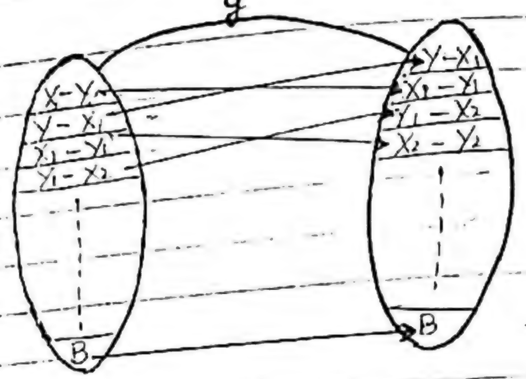
Here X_0 means X and Y_0 means Y.

Now define

$g: X \to Y$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in X_i - Y_i \\ x & \text{if } x \in Y_i - X_{i+1} \text{ or } B \end{cases}$$

i.e.



Obviously g is bijective
$\Rightarrow X \sim Y$.

COROLLARY:-

If $X \precsim Y$ and $Y \precsim X$ then $X \sim Y$

PROOF:-

As $X \precsim Y$ so X is equivalent some subset of Y say Y_1 s.t $X \sim Y_1 \subseteq Y$ — (i)

and $Y \precsim X \Rightarrow Y \sim X_1 \subseteq X \longrightarrow$ (ii)

Since $Y_1 \subseteq Y$ and $Y \sim X_1$, so then there exist a subset Y_2 of X_1 s.t $Y_1 \sim X_2$ i.e. $Y_1 \sim Y_2 \subseteq X_1$

from (i)

$X \sim Y_1$ and $Y_1 \sim Y_2$ so $X \sim Y_2$

Also $Y_2 \subseteq X_1 \subseteq X$ and $X \sim Y_2$

So by Schroder Bernstein Theorem
$X \sim X_1$

So by (ii) $Y \sim X_1$ and Now $X_1 \sim X$
$\Rightarrow Y \sim X \Rightarrow X \sim Y$.

COROLLARY:-

If $\alpha \leq \beta$ and $\beta \leq \alpha$ Then $\alpha = \beta$

PROOF:-

Let $\alpha = \#(X)$ and $\beta = \#(Y)$

Then $\alpha \leq \beta \Rightarrow X \precsim Y$

$\beta \leq \alpha \Rightarrow Y \precsim X$

Then $X \sim Y \Rightarrow \#(X) = \#(Y)$

$\Rightarrow \alpha = \beta.$

EXPONENT OF CARDINAL NUMBERS:-

If A and B are two non empty sets then B^A denotes the collection of all functions from A to B.

If $\#(A) = \alpha$ and $\#(B) = \beta$ then $\#(B^A) = \beta^\alpha$

CANTOR'S THEOREM:-

For any set A, $A \prec 2^A$

PROOF:-

Define $f: A \to 2^A$ by

$f(a) = \{a\}$

Then $a \neq b \Rightarrow \{a\} \neq \{b\}$

$\Rightarrow f(a) \neq f(b)$

$\Rightarrow f$ is 1-1.

$\Rightarrow A \lesssim 2^A$

Now we prove that $A \nsim 2^A$

Suppose on the contrary that $A \sim 2^A$ then there exists a bijective function $g: A \to 2^A$.

Now, consider a subset B of A as

$B = \{x : x \in A$ and $x \notin g(x)\}$

Then as g is bijective so, g is also onto, then for $B \in 2^A$ there exist some $b \in A$ s.t $g(b) = B$

Now, if $b \in B \Rightarrow b \notin g(b) \Rightarrow b \notin B$

if $b \notin B \Rightarrow b \in g(b) \Rightarrow b \in B$

which is a contradiction so our supposition is wrong and hence $A \nsim 2^A$

$\Rightarrow A < 2^A$

THEOREM:-

For any cardinals α, β, γ

$$\alpha^{\beta} \alpha^{\gamma} = \alpha^{\beta+\gamma}$$

PROOF:-

let A, B and C be three pairwise disjoint sets and $\#(A) = \alpha$, $\#(B) = \beta$ and $\#(C) = \gamma$

then $\alpha^{\beta} \cdot \alpha^{\gamma} = \#(A^B \times A^C)$ and

$$\alpha^{\beta+\gamma} = \#(A^{B \cup C})$$

To prove $\alpha^{\beta}.\alpha^{\gamma} = \alpha^{\beta+\gamma}$ we will have to prove $A^B \times A^C \sim A^{B\cup C}$

Let $f \in A^{B\cup C}$ then $f : B\cup C \to A$
then there exists
$f_B : B \to A$ and $f_C : C \to A$
$\Rightarrow f_B \in A^B$ and $f_C \in A^C$
$\Rightarrow (f_B, f_C) \in A^B \times A^C$
$\Rightarrow$ for every element in $A^{B\cup C}$ we have an element $A^B \times A^C$ $\longrightarrow$ (i)
Now Let $(f, g) \in A^B \times A^C$
$\Rightarrow f \in A^B$ and $g \in A^C$
$\Rightarrow f : B \to A$ and $g : C \to A$ then There exist $h : B\cup C \to A$ which is extension of both f and g.
$(\because B \cap C = \phi)$
$\Rightarrow h \in A^{B\cup C}$
So for every element in $A^B \times A^C$, we have an element $A^{B\cup C}$ $\longrightarrow$ (ii)
Now define
$F : A^B \times A^C \to A^{B\cup C}$ by
$F((f, g)) = h$, where h is extension of f and g.
Then f is bijective
$\Rightarrow A^B \times A^C \sim A^{B\cup C}$
$\Rightarrow \alpha^{\beta}.\alpha^{\gamma} = \alpha^{\beta+\gamma}$

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QUESTION :- If $\#(B) = 2$ Then $\#(2^A) = \#(B^A)$ for any set A.

SOLUTION:-

Let $B = \{0, 1\}$ Then $\#(B) = 2$

and $B^A = C(A)$

$\Rightarrow B^A \sim C(A)$

But $C(A) \sim 2^A$

$\Rightarrow B^A \sim 2^A$

$\Rightarrow \#(B^A) = \#(2^A)$

QUESTION:-

If $\alpha \leq \beta$ Then there exists a set B with some subset A, such that $\#(A) = \alpha$ and $\#(B) = \beta$.

SOLUTION:-

Let B be a set with $\#(B) = \beta$

This is obvious in either case B is denumerable, nondenumerable or finite.

Case I:-

Now, If B is non denumerable then $\beta = c$

Then $\alpha \leq c \Rightarrow$ either $\alpha < c$ or $\alpha = c$

if $\alpha = c$, Then B itself is the set s.t

$\#(B) = \alpha$ and $B \subseteq B$

if $\alpha < c$, Then either α is finite cardinal

or α is cardinality of some denumerable set.

If α is finite cardinal, then there exists some finite subset A of B s.t $\#(A) = \alpha$

If α is cardinality of some denumerable set, then as for every non denumerable set has a denumerable subset then, there exist a denumerable subset A of B s.t $\#(A) = \alpha$.

Case II :-

If B is some denumerable set, then $\beta = a$. Then $\alpha \leq \beta$ means $\alpha \leq a$

$\Rightarrow \alpha = a$ or $\alpha < a$.

If $\alpha = a$, then B itself is the set with $\#(B) = \alpha$ and $B \subseteq B$.

If $\alpha < a$, then α is some finite cardinal, then there exists some finite subset A of B s.t $\#(A) = \alpha$

Case III :-

If B is some finite set and $\#(B) = \beta$

Then β is some natural numbers.

Then, $\alpha \leq B \Rightarrow \alpha = \beta$ or $\alpha < \beta$

If $\alpha = \beta \Rightarrow B$ itself is the set s.t

$\#(B) = \alpha$ and $B \subset B$

If $\alpha < \beta$, then α is some finite cardinal. Then, there exist some finite subset A of B, s.t. $\#(A) = \alpha$

QUESTION:-

If α, β, γ are cardinals and $\alpha \le \beta$,
Then
(i) $\alpha + \gamma \le \beta + \gamma$
(ii) $\alpha\gamma \le \beta\gamma$
(iii) $\alpha^{\gamma} = \beta^{\gamma}$
(iv) $\gamma^{\alpha} \le \gamma^{\beta}$

SOLUTION:-

Let A, B, C be the three sets, s.t
$\#(A) = \alpha$, $\#(B) = \beta$ and $\#(C) = \gamma$,
$A \subseteq B$ and $B \cap C = \phi$.

(i) Now $\alpha + \gamma = \#(A \cup C)$ and $\beta + \gamma = \#(B \cup C)$
As, $A \subseteq B \Rightarrow A \cup C \subseteq B \cup C$
$\Rightarrow \#(A \cup C) \le \#(B \cup C)$
$\Rightarrow \alpha + \gamma \le \beta + \gamma$

(ii) Now $A \subseteq B \Rightarrow A \times C \subseteq B \times C$
$\Rightarrow \#(A \times C) \le \#(B \times C)$
$\Rightarrow \alpha\gamma \le \beta\gamma$

(iii) Now $\#(A^C) = \alpha^{\gamma}$ and $\#(B^C) = \beta^{\gamma}$
Now let $f \in A^C$
$\Rightarrow f : C \to A$
$\Rightarrow f : C \to B$ is a function
$\because A \subseteq B$
$\Rightarrow f \in B^C$
$\Rightarrow A^C \subseteq B^C$

$\Rightarrow \#(A^c) \leq \#(B^c)$
$\Rightarrow \alpha^\gamma \leq \beta^\gamma$

(iv) $\gamma^\alpha \leq \gamma^\beta$

Now $\gamma^\alpha = \#(C^A)$ and $\gamma^\beta = \#(C^B)$

let $f \in C^A \Rightarrow f : A \to C$

Now as $A \subseteq B$, so let $f^* : B \to C$ be its extension on B.

$\Rightarrow f^* \in C^B$

Now define, $F : C^A \to C^B$ by
$F(f) = f^*$

Then, if $f \neq g$, $f, g \in C^A$
then $f^* \neq g^*$
$\Rightarrow F(f) \neq F(g)$
$\Rightarrow F$ is 1-1
$\Rightarrow \#(C^A) \leq \#(C^B)$
$\Rightarrow \gamma^\alpha \leq \gamma^\beta$

QUESTION:-

Show that set T of all Transidential numbers has cardinality c.

SOLUTION:-

Let A be the set of all algebraic numbers then $A \cup T = \mathbb{R}$

We know that A is denumerable

To prove T is non-denumerable.

Suppose on the contrary T is denumerable, Then

$A \cup T = \mathbb{R}$ is denumerable

A contradiction, so our supposition is wrong and hence T is non denumerable.

$\Rightarrow \#(T) = c$

If any set A has cardinality c, then A has the power of continuum.

2017 Question:- Prove that union of a countable family of countable sets is countable.

Solution:- Let

γ: be an infinite countable family of infinite countable pairwise disjoint sets.

γ_1: be any other possible type of countable family of countable sets. Then obviously $\cup\gamma_1 \subseteq \cup\gamma$. So if $\cup\gamma$ is countable, then $\cup\gamma_1$ is countable.

So we now prove only that $\cup\gamma$ is countable.

Let $\gamma = \{A_i\}_{i=1}^{\infty} = \{A_1, A_2, A_3, \ldots\}$

where for each "i"

$A_i = \{a_{i1}, a_{i2}, a_{i3}, \ldots\}$

Then $\bigcup_{i=1}^{\infty} A_i = \cup\gamma = \{ a_{11}, a_{12}, a_{13}, \ldots,$
$a_{21}, a_{22}, a_{23}, \ldots,$
$a_{31}, a_{32}, a_{33}, \ldots,$
$\vdots$
$\}$

Define $f : U\gamma \longrightarrow N \times N$ by
$f(a_{ij}) = (i,j)$ then f is bijective (proved already).
$\Rightarrow U\gamma$ is denumerable.
$\Rightarrow U\gamma$ is countable.

Question:.
If A & B are two sets such that A is equivalent to a subset of B & B is equivalent to a subset of B, then $A \sim B$.

Solution.
By given condition
$A \sim B_1 \subseteq B$ & $B \sim A_1 \subseteq A$.
Now as $B_1 \subseteq B$ & $B \sim A_1$ so then there exists $B_2 \subseteq A_1$ such that
$B_1 \sim B_2 \subseteq A_1$
Now $A \sim B_1$ & $B_1 \sim B_2$, so $A \sim B_2$
As $B_2 \subset A_1 \subseteq A$ & $A \sim B_2$, so by Schroder Bernstein Theorem
$A \sim A_1$.
Now $A \sim A_1$ & $A_1 \sim B$
So $A \sim B$.

Question:- Prove that $\aleph_0 + \beta = \beta$ where $\aleph_0 = \#(N)$ & β is any transfinite cardinal.

Solution:- This is same $\alpha + \beta = \beta$, the only difference is that "α" is replaced and denoted by $\aleph_0$.

Question: Show that $N \times N \sim N$.

Solution:- As N is denumerable so $N \times N$ is denumerable $\Rightarrow N \times N \sim N$.

Question:- If λ is any infinite cardinal, then $\lambda+\lambda=\lambda$

Solution:- By continuum Hypothesis, either λ is "a" or "c". Further $\lambda+\lambda=\lambda$ follows by the fact that union of two denumerable sets is denumerable and also the union of two non-denumerable sets is non-denumerable.

Question:- Prove that $\mathbb{R}^2 \sim \mathbb{R}$ & hence $\mathbb{R}^n \sim \mathbb{R}$

Solution:- Consider $S =]0,1[= \{x \in \mathbb{R} : 0<x<1\}$ & $S^2 = S \times S = \{(x,y) : 0<x<1 \text{ \& } 0<y<1\}$.

Now for $(x,y) \in S^2 \Rightarrow x, y \in S =]0,1[$ can be written in decimal form

$x = 0.d_1d_2d_3\cdots\cdots$, $y = 0.e_1e_2e_3\cdots\cdots$ where all $d_i, e_i \in \{0,1,2,\cdots,9\}$ & each decimal expansion contains infinite number of non-zero digits e.g. $\frac{1}{2}$ is written $0.4999\cdots$ instead of $0.50000\cdots$. Now define $f : S^2 \to S$ by

$f(x,y) = f(0.d_1d_2d_3\cdots, 0.e_1e_2e_3\cdots) = 0.d_1e_1d_2e_2\cdots$

Then due to uniqueness of decimal expansion f is $1-1 \Rightarrow S^2 \precsim S$.

Next define

$f : S \to S^2$ by

$f(x) = (x, 0.5)$ then also f is $1-1$. Thus $S \precsim S^2$. Then by S.B Theorem $S^2 \sim S$

Now $\mathbb{R} \sim S \Rightarrow \mathbb{R}^2 = \mathbb{R}\times\mathbb{R} \sim S \times S = S^2 \sim S \sim \mathbb{R} \Rightarrow \mathbb{R}^2 \sim \mathbb{R}$.

Next $\mathbb{R}^3 \sim \mathbb{R}^2 \times \mathbb{R} \sim \mathbb{R}\times\mathbb{R} \sim \mathbb{R}$

Similarly by induction

$\mathbb{R}^n \sim \mathbb{R}$.

CHAPTER # 02

PARTIALLY AND TOTALLY ORDERED SETS

Let A and B be two non empty sets, then cartesian product of A and B is denoted and defined as:

$$A \times B = \{(a,b) : a \in A \wedge b \in B\}$$

BINARY RELATION:- Any subset of $A \times B$ is said to be binary relation from A to B. It is usually denoted by R.

Any subset R of $A \times A$ is said to be binary relation or simply relation on A instead of "from A to A"

EXAMPLE:-

Let $A = \{1,2,3,4,5,6,7,8\}$, $B = \{1,2,3,4\}$ then,

$R_1 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\} \subseteq B \times B$

$R_2 = \{(1,1),(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\} \subseteq B \times B$

$R_3 = \{(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(1,8),(2,2),(2,4),(2,6),(2,8),(3,3),(3,6),(4,4),(4,8),(5,5),(6,6),(7,7),(8,8)\} \subseteq A \times A$

Then, R_1 and R_2 are binary relations on B and R_3 is a binary relation on A.

Note that one can also write

$R_1 = \{(x,y) \in B \times B : x \leq y\}$

$R_2 = \{(x,y) \in B \times B : x < y\}$
$R_3 = \{(x,y) \in A \times A : x \text{ divides } y\}$

If R is a relation on A and ordered pair $(x,y) \in R$, Then we write xRy and read it as "x is related to y"

DEFINITION:-

A relation R on a set X is said to be

(i) Reflexive :- If for all $x \in X$, xRx.

(ii) Antisymmetric : if for $x, y \in X$, xRy and yRx then $x = y$.

(iii) Transitive: if for $x, y, z \in X$, xRy and yRz then xRz.

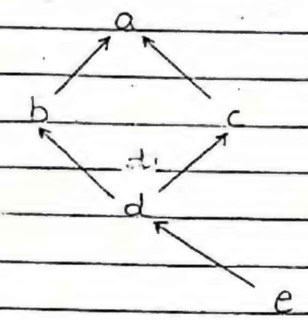
A relation R on X is said to be partial order relation on X if it is reflexive, anti-symmetric and transitive.

EXAMPLES:-

(i) R_1 and R_3 defined above are partial order relations.

(ii) let A be any collection of sets and for $X, Y \in A$, XRY, if $X \subseteq Y$.
Then R is partial order relation on A.

(iii) let $W = \{a, b, c, d, e\}$ and consir the diagram.



where for $x, y \in W$, xRy if either $x = y$ or one can go from x to y through indicated direction.

Then, this relation defines partial order on W

REMARKS:-

(i) If xRy then we write $x \preceq y$ and read it as x "preceeds" y.

(ii) Antisymmetric relation can also be defined "A relation R on X is antisymmetric when any two distinct element a and b of X, either aRb or bRa

(iii) If $x \preceq y$ i.e x preceeds y, then we say y dominates x.

(iv) $x \prec y$ means x strictly preceeds y i.e x preceeds y but $x \neq y$.

(v) $x \not\preceq y$ means x does not preceeds y.

(vi) If R is partial order on A, then we say A is partial order set (poset) with order R or simply (A,R) is poset.

(vii) If R is partial order of A, then R^{-1}, inverse relation of R is also partial order on A.

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TOTALLY ORDERED SET:-

If (A,R) is a partial order set then for $a, b \in A$ s.t $a \neq b$, we have no guarantee that $a \precsim b$ or $b \precsim a$.

e.g.

$A = \{1, 2, 3, 4, 5\}$ and $x, y \in A$, $x \precsim y$ when x divides y. Then A is partial order set

Let $2, 3 \in A$, $2 \neq 3$ $2 \not\precsim 3$ and $3 \not\precsim 2$

Now consider,

$B = \{1, 2, 4, 8, 16, 32, 64\}$

and for $x, y \in B$ s.t. $x \neq y$, $x \precsim y$ when x divides y. Then, In set B under this partial order relation for any $a, b \in B$ s.t $a \neq b$ either $a \precsim b$ or $b \precsim a$

Hence, A partial order on a set X is a total order if for $a, b \in X$ either $a \precsim b$ or $b \precsim a$

NOTE:-

If either $a \preceq b$ or $b \preceq a$ then we say a and b are compareable. So in a poset it is not necessary that every two elements are compareable, but in a total ordered set every two elements are compareable.

REMARK:-

A set X is said to be ordered set if it is either partial ordered set or totally ordered set.

SUBSET OF AN ORDERED SET:-

Let (A, R) be an ordered set and $B \subset A$. Then R defines a partial order R' on B in the way for $x, y \in B$, $xR'y$ if xRy. From this we can say every subset of an ordered set is again an ordered set.

e.g.

$A = \{1, 2, 3, 4\}$ and $x, y \in A$, xRy when $x \leq y$

Then,

$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$

and $B = \{1, 3\} \subset A$

$R' = \{(1,1), (1,3), (3,3)\}$

REMARK:-

Every subset of a totally ordered set is totally ordered but converse is not true in general e.g; $A = \{1, 2, 3, \ldots, 8\}$

and $x, y \in A$, $x \precsim y$ if x divides y then, A is not totally ordered But $B_1 = \{1, 2, 4, 8\}$ and $B_2 = \{1, 3, 6\}$ are totally ordered subset of A.

Definitions:-

let A be an ordered set then,

(i) An element $a \in A$ is said to be first element of A if for every element $x \in A$, $a \precsim x$ or every element $x \in A$ dominates a.

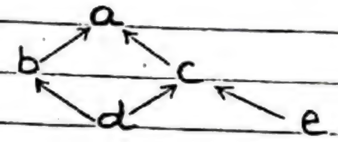
(ii) An element $b \in A$ is said to be last element of A if for every element $x \in A$, $x \precsim b$ or b dominates every element x of A

(iii) An element $m \in A$ is said to be minimal element of A if, any element $x \in A$ preceeds m then $x = m$.
In other words $m \in A$ is said to be minimal element of A, if no other element of A preceeds m

(iv) An element m' of A is said to be maximal element of A if for $x \in A$, $m' \precsim x \Rightarrow m' = x$
or $m' \in A$ is said to be maximal element of A if no other element of A dominates m'

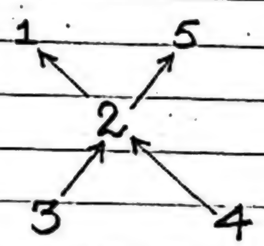
EXAMPLE:-

Let $W = \{a, b, c, d, e\}$ and



Set has no first element.
a is last element of W.
d and e are two minimal elements of W
a is the maximal element of W.

EXAMPLE:- Let $A = \{1, 2, 3, 4, 5\}$ and



A has no first element.
A has no last element.
1 and 5 are maximal elements of W.
3 and 4 are minimal element of W.

EXAMPLE:-

Let $A = \{x \in R : 0 < x < 1\}$ and $a, b \in A$,
$a \precsim b$ if $a \leq b$.

A has no first element, no last

element, no maximal element and no minimal element.

Remarks:-

(i) If 'a' is the first element of set A then 'a' is minimal element of A and it is then the only minimal element of A.

If b is last element of A, then b is maximal element of A and it is then only maximal element of A.

Converse of this remark is not true in general.

(ii) In a totally ordered set first element and minimal element coincides and maximal element and last element coincides, Provided they exist.

(iii) every finite partially ordered set has at least one maximal element and at least one minimal element. This does not hold for infinite set.

THEOREM:- In an ordered set first and last elements are unique.

PROOF:- Suppose on the contrary that "a" & "b" are two 1st elements of an ordered set A. Then for all $x \in A$, $a \precsim x$ & $b \precsim x$.
As $a \precsim x$, for all x, so $a \precsim b$.
As $b \precsim x$, for all x, so $b \precsim a$.
As $a \precsim b$ and $b \precsim a$ so then $a = b$.
A contradiction, so our supposition is wrong, hence 1st element is unique.

Suppose on the contrary that a & b are two last elements of an ordered set A. Then for all $x \in A$, $x \precsim a$ and $x \precsim b$.
As for all x, $x \precsim a$, so $b \precsim a$.
As for all x, $x \precsim b$, so

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THEOREM:- In an ordered set if 'a' is the first element then 'a' is the only minimal element.

PROOF:-

let A be an ordered set and $a \in A$ be its first element then of course 'a' is minimal element.
To prove 'a' is the only minimal element of A.
Suppose, on the contrary that 'a′' is another minimal element of A.
Now as 'a' is first element of A, so $a \preceq x$ for all $x \in A$.
Then also $a \preceq a' \Rightarrow a = a'$ $\because$ a' is the minimal element of A.

$\Rightarrow$ a is the only minimal element of A.

THEOREM:- In an ordered set last element is the only maximal element.

PROOF:-

let A be an ordered set and $a \in A$ be its last element, then of course

a is the maximal element of A.
To prove 'a' is the only maximal element of A.

Suppose on the contrary that m' is another maximal element of A. Now as 'a' is the last element of A, so $x \precsim a$ for all $x \in A$.
Then also

$$m' \precsim a$$
$$\Rightarrow m' = a$$
$$\left(\because m' \text{ is the maximal element of } A \right)$$

$\Rightarrow a$ is the only maximal element of A

THEOREM:- In a totally ordered set no two maximal (or minimal) element exists.

PROOF:- Let A be a totally ordered set with $a, b \in A$ be two maximal elements of A. As A is totally ordered, so
(i) $a \precsim b$ or (ii) $b \precsim a$ or (iii) $a = b$
If $a \precsim b$ & a is maximal, so $a = b$
If $b \precsim a$ & as b is maximal, so $b = a$
If $a = b$, then nothing to prove.
So in either case we have $a = b$.
So no two maximal elements exists.

Now suppose $a, b \in A$ are two minimal elements of A. Since A is totally ordered so either $a \precsim b$ or $b \precsim a$.
If $a \precsim b$, then $a = b$, as b is minimal.
If $b \precsim a$, then $b = a$, as a is minimal.
Hence in either case $a = b$. Thus no two minimal elements exists.

THEOREM:- Every finite partially ordered set has at least one minimal element.

PROOF:- Let A be a finite partially ordered set. Let $a_1 \in A$. If a_1 is the minimal element, then theorem is proved.
If a_1 is not minimal element then there is $a_2 \in A$ s.t $a_2 \precsim a_1$.
If a_2 is minimal element, then Theorem is proved.
If a_2 is not minimal element then there is $a_3 \in A$ s.t. $a_3 \precsim a_2$ and continuing this we have.

$$\ldots \precsim a_{i+1} \precsim a_i \ldots \precsim a_3 \precsim a_2 \precsim a_1$$

Since A is finite, so this sequence must terminates at some a_n i.e.

$$a_n \precsim a_{n-1} \precsim \ldots \precsim a_{i+1} \precsim a_i \precsim \ldots \precsim a_2 \precsim a_1$$

and no other element of A preceeds a_n then a_n is the minimal element of A.

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THEOREM:- Every finite partially ordered set have at least one maximal element.

PROOF:- Let A be any finite partially ordered set. Let $a_1 \in A$, If a_1 is maximal element then theorem is proved.
If a_1 is not maximal element then there is $a_2 \in A$ s.t $a_1 \precsim a_2$.
If a_2 is maximal element then theorem is proved. If a_2 is not maximal element then there is $a_3 \in A$ s.t $a_2 \precsim a_3$
By continuing this we have

$$a_1 \precsim a_2 \precsim a_3 \precsim \dots \precsim a_i \precsim a_{i+1} \precsim \dots$$

Since A is finite, so then this sequence must terminates at some a_n.
i.e.

$$a_1 \precsim a_2 \precsim a_3 \precsim \dots \precsim a_i \precsim a_{i+1} \dots \precsim a_n$$

and no other element of A dominates a_n.
$\Rightarrow$ a_n is the maximal element of A.

Definition :-
Let A be an ordered set and $B \subseteq A$ then an element
(i) $l \in A$ is said to be lower bound of B, if for all $x \in B$, $l \precsim x$

(ii) $U \in A$ is said to be upper bound of B if for all $x \in B$, $x \preceq u$.

Definition :-

Let A be an ordered set and $B \subseteq A$ then,

(i) A lower bound of B is said to be greatest lower bound or infimum of B if it dominates all other lower bounds of B.
It is denoted by $\text{Inf}(B)$

(ii) An upper bound of B is said to be least upper bound of B if it preceeds all other upper bounds of B.
It is denoted by $\text{Sup}(B)$.

Definition :-

A Set B of an ordered set A is said to be

(i) Bounded below if it has lower bound in A.

(ii) Bounded above if it has upper bound in A.

(iii) Bounded if it is both bounded below and bounded above.

EXAMPLE:-

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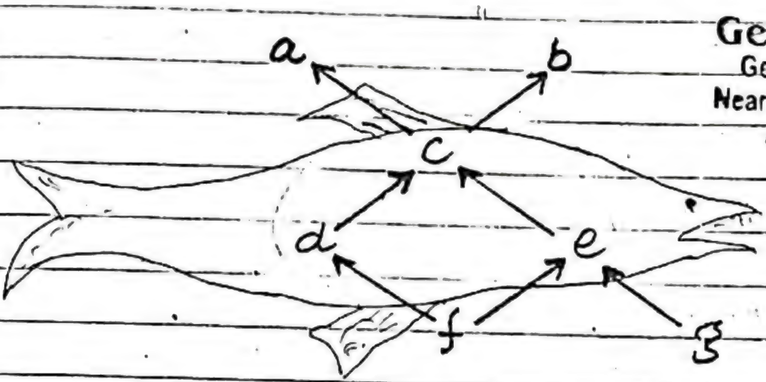
Let $A = \{a, b, c, d, e, f, g\}$ and $B = \{c, d, e\}$ and order in A

is defined by the directed diagram



Then f is lower bound of B
a, b, c are upper bounds of B
$Inf(B) = f \notin B$ and $Sup(B) = c \in B$

EXAMPLE 2:-

Let A be any bounded subset of $\mathbb{R}$, then under the natural order $Inf(A)$ and $Sup(A)$ exists.

EXAMPLE 3:-

Let A be set Q of rational number and $B = \{x : x \in Q \wedge 2 < x^2 < 3\}$
Then B is bounded subset of Q, but B has no suprimum and no infimum.

REMARK:-

Let $\{A_i\}_{i \in I}$ be a totally ordered family (i.e. I is totally ordered set) of pairwise disjoint totally ordered sets then, $\bigcup_{i \in I} A_i$ is totally ordered

(unless otherwise stated) as follows:

Let $a, b \in \bigcup_{i \in I} A_i$, hence there exist $j, k \in I$, s.t $a \in A_j$ and $b \in A_k$.

Now if $j \lneq k$ then $a \lneq b$ and if $j = k$ and a, b are ordered by the ordering of A_j.

EXAMPLE:-

Let $A_1 = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ be a totally ordered set with $a_i \lneq a_j$ iff $i \lneq j$

and $A_2 = \{1, 2, 3, \ldots\} \subseteq \mathbb{R}$ is also a totally ordered set with natural order then,

$A_1 \cup A_2 = \{A_1 ; A_2\} = \{a_1, a_2, \ldots, a_n, \ldots, 1, 2, 3, \ldots\}$

Q. In the set $A = \{2, 3, 4, 5, 6\}$ define the relation $m \leq n$ to mean that m divides n, show that A is partially ordered set. Determine maximal & minimal elements of A.

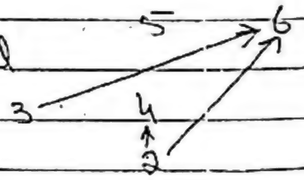
Solution:- * As for all $m \in A$, m divides m, so $m \leq m \Rightarrow \leq$ is reflexive.

* As for $m, n \in A$, if $m \leq n$ & $n \leq m \Rightarrow m$ divides n & n divides $m \Rightarrow m = n$, $\leq$ is antisymmetric

* $\leq$ is transitive. $\Rightarrow (A, \leq)$ is POSET.

2, 3, 5 are minimal elements of A.

4, 5, 6 are maximal elements of A.



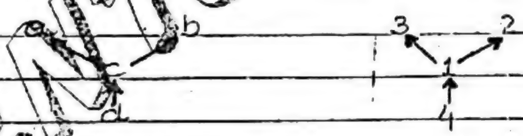
SIMILAR SETS

Two ordered sets A and B are said to be similar sets if there exist a bijective mapping $f: A \to B$ which satisfies the property:

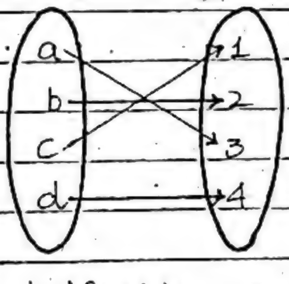
for $a, a' \in A$, $a \lesssim a'$ iff $f(a) \lesssim f(a')$. Then the function f is called similar mapping.

EXAMPLES:-

(i) Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3, 4\}$ and ordered in A and B is defined by the diagram



Then $f: A \to B$ is given by



is a similarity mapping.

(ii) Let $N = \{1, 2, 3, \ldots\}$ and $E = \{2, 4, 6, \ldots\}$. Then N and E are ordered set under natural order. Then $F: N \to E$ defined by $f(n) = 2n$ is a similarity mapping.

(iii) let $N = \{1, 2, 3, \ldots\}$ and $M = \{-1, -2, -3, \ldots\}$
Then N and M are ordered set under natural order. Then N and M are not similar Because otherwise if so, then there is $f: N \to M$ is a similarity mapping. Now as for every $a \in N$, $1 \precsim a$
$\Rightarrow f(1) \precsim f(a)$.
$\Rightarrow f(1)$ is a first element in M which is a contradiction, because in M, first element does not exist. So no similarity mapping between N and M exists.

REMARK:-
If A and B are similar then we write $A \simeq B$

THEOREM:- If A is totally ordered set and $B \simeq A$, then B is also totally ordered set.

PROOF:-
As $B \simeq A$, so let $f: B \to A$ be a similarity mapping. To prove B is totally ordered. Suppose on the contrary B is not totally ordered. Then there exist $b_1, b_2 \in B$ such that b_1 and b_2 are not compareable i.e. $b_1 \not\precsim b_2$ and $b_2 \not\precsim b_1$
Then $f(b_1) \not\precsim f(b_2)$ and $f(b_2) \not\precsim f(b_1)$
But $b_1, b_2 \in B \Rightarrow f(b_1), f(b_2) \in A$

and $f(b_1), f(b_2)$ are not compareable.
$\Rightarrow$ A is not totally ordered which is a contradiction, so our supposition is wrong and hence B is totally ordered.

REMARK:-

If A is similar to B, then In particular A is equivalent to B. Converse of this is not true in general because $N \sim M$ where $M = \{-1, -2, -3, \ldots\}$ by $f(n) = -n$
But $N \not\simeq M$.

THEOREM:-

The relation of similarity is an equivalence relation.

PROOF:-

REFLEXIVE:-

let A be any ordered set then $I: A \to A$ defined by $I(x) = x$ is a similar mapping.

$\Rightarrow A \simeq B$

$\Rightarrow \simeq$ is reflexive.

SYMMETRIC:-

let $A \sim B$ then there exist $f: A \to B$ a similarity mapping. As $f: A \to B$ is bijective so, $f^{-1}: B \to A$ exists and bijective.

Now we prove $f^{-1}: B \to A$ is also similarity mapping.
Now as $f: A \to B$ is similarity mapping and so is bijection, then f is onto, so there exists $a_1, a_2 \in A$ s.t.
$f(a_1) = b_1$ and $f(a_2) = b_2$ Then, as
$b_1 \precsim b_2$ means $f(a_1) \precsim f(a_2)$
As f is similarity mapping, so $a_1 \precsim a_2$
$\Rightarrow f^{-1}(b_1) \precsim f^{-1}(b_2)$

$\because f(a_1) = b_1 \Rightarrow f^{-1}(b_1) = a_1$
& $f(a_2) = b_2 \Rightarrow f^{-1}(b_2) = a_2$

Hence obvious, $b_1 \precsim b_2$ iff $f^{-1}(b_1) \precsim f^{-1}(b_2)$
$\Rightarrow f^{-1}: B \to A$ is similarity mapping.

TRANSITIVE:
Let $A \simeq B$ and $B \simeq C$, Then there exists $f: A \to B$ and $g: B \to C$ be similarity mapping. As f, g are bijection so $gof: A \to C$ is also bijection.
Now, we prove gof is similarity mapping.
let $a_1, a_2 \in A$ s.t $a_1 \precsim a_2$
Now $a_1 \precsim a_2$ iff $f(a_1) \precsim f(a_2)$
iff $g(f(a_1)) \precsim g(f(a_2))$
$\Leftrightarrow (gof)(a_1) \precsim (gof)(a_2)$

$\Rightarrow gof: A \to C$ is similarity mapping.
$\Rightarrow A \sim C \Rightarrow \sim$ is transitive.

Hence similarity of sets is an equivalance relation.

THEOREM:- If $f: A \to B$ is a similarity mapp and a is first element (last element) of A iff $f(a)$ is first (last) element of B.

PROOF:-

Let us assume a is first element of A, then for all $x \in A \Rightarrow a \precsim x$

$\Rightarrow f(a) \precsim f(x)$, for all $f(x) \in B$

$\Rightarrow f(a)$ is first element of B.

Conversely assume $f(a)$ is first element of B then $f(a) \precsim y$, for all $y \in B$. As $f: A \to B$ is bijection so, for all $y \in B$, we have $x \in A$ s.t. $f(x) = y$.

$$\Rightarrow f(a) \precsim f(x) \quad , \forall \ f(x) \in B$$

$$\Rightarrow a \precsim x \quad ; \forall \ x \in A$$

a is first element of A.

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SORRY

THEOREM:- If $f: A \to B$ is a similarity mapping then $a \in A$ is the minimal (maximal) element of A iff $f(a)$ is minimal (maximal) element of B.

PROOF:-

Let us assume $a \in A$ is minimal element of A, then for all $x \in A$, other then a, $x \not\prec a$. As $f: A \to B$ is similarity mapping so then $f(x) \not\prec f(a)$. Since f is similarity mapping, so it is also bijective and due to this reason we have for all $y \in B$, there exist $x \in A$ such that

$f(x) = y \Rightarrow$ for all $y \in B$, $y \not\prec f(a)$

$\Rightarrow f(a)$ is minimal element of B.

Conversely assume $f(a) \in B$ is minimal element of B.

Now, As $f: A \to B$ is similarity mapp so $f^{-1}: B \to A$ is also similarity mapp
$\Rightarrow f^{-1}: B \to A$ is a similarity mapp and $f(a) \in B$ is its minimal element then, by above discussion (i.e. indirect part before conversely) $f^{-1}(f(a)) = a$ is minimal element of A.

(12) WELL ORDERED SETS AND ORDINAL NUBERS

An ordered set A is said to be well ordered if its every subset has first element

EXAMPLE:-

(1). Set $\mathbb{N}$ of natural number is well ordered under natural order i.e '$\leq$'

(2). Note that under '$\leq$' sets $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are not well ordered sets.

(3). For any non empty set, X, $P(X)$ is not necessarily well ordered under '$\leq$'

THEOREM:-

Every well ordered set is totally ordered.

PROOF:-

let A be a well ordered set.

To prove A is totally ordered.

let $a, b \in A \Rightarrow \{a, b\} \subseteq A$.

$\Rightarrow \{a, b\}$ has first element.

$\Rightarrow$ either a is the first element of $\{a, b\}$ or b is the first element of $\{a, b\}$.

$\Rightarrow$ either $a \preceq b$ or $b \preceq a$

$\Rightarrow A$ is totally ordered.

THEOREM:- Every subset of a well ordered set is well ordered.

PROOF:-

let 'A' be a well ordered set and $B \subseteq A$, To prove B is well ordered.

let $C \subseteq B \Rightarrow C \subseteq A$

$\Rightarrow C$ has first element.

(because A is well ordered)

$\Rightarrow$ B is well ordered

PRINCIPAL OF MATHEMATICAL INDUCTION

let $S \subseteq N$ with the properties:

(i) $1 \in S$

(ii) for $n \in S \Rightarrow n+1 \in S$ Then $S = N$.

INITIAL SEGMENTS: let A be a well ordered set and $a \in A$, Then initial segment of a is denoted and defined as:

$S(a) = \{x : x \in A \text{ and } x < a\}$

Note that $a \notin S(a)$

EXAMPLE:-

(1). let $n \in N$ and under natural order '$\leq$', $S(n) = \{1, 2, 3, \ldots, n-1\}$

(2) let $A = \{1, 3, 5, \ldots\}$ and $B = \{2, 4, 6, \ldots\}$ Then both A and B are well ordered under natural order. Then

$\{A ; B\} = \{1, 3, 5, \ldots, 2, 4, 6, \ldots\}$

Note that $S(5) = \{1, 3\}$

$S(2) = \{1, 3, 5, \ldots\}$ and $S(4) = \{1, 3, 5, \ldots, 2\}$

2015

Principal of Transfinite Induction:-

let A be a well ordered set and $S \subseteq A$ with the properties

i. $a_0 \in S$ and a_0 is 1st element of A
ii. $S(a) \subseteq S \Rightarrow a \in S$ then $S = A$.

Proof:-

Suppose on the contrary that $S \neq A$
Then $S \subset A \Rightarrow A \setminus S \neq \phi$
As $A \setminus S \subseteq A$ and A is well ordered
$\Rightarrow A \setminus S$ has first element.

let $t_0 \in A \setminus S$ be its first element
Now, consider $S(t_0) = \{x : x \in A \text{ and } x < t_0\}$
let $x \in S(t_0) \Rightarrow x < t_0$
$\Rightarrow x \notin A \setminus S$
($\because t_0 \in A \setminus S$ is its first element)

$\Rightarrow x \in S$
$\Rightarrow S(t_0) \subseteq S$
$\Rightarrow t_0 \in S$
by given 2nd condition
$\Rightarrow t_0 \notin A \setminus S$

$\Rightarrow$ A contradiction because t_0 is first element of $A \setminus S$.

So our supposition is wrong.
Hence

$$S = A$$

IMMEDIATE SUCCESSOR:-

An element $a \in A$ is said to be immediate successor of an element $b \in A$ if $b \prec a$ and there is no other element $c \in A$ s.t. $b \prec c \prec a$.

IMMEDIATE PREDECESSOR

If an element $a \in A$ is immediate successor of $b \in A$, then b is called immediate predecessor of a.

2016

THEOREM:- Let A be a well ordered set and $B \subseteq A$ let $f: A \to B$ be a similarity mapping Then for all $x \in A$, $x \precsim f(x)$.

PROOF:-

Let $D = \{x \in A : f(x) \prec x\}$

If theorem is true i.e. for all $x \in A$, $x \precsim f(x)$, then D is an empty set.

Suppose on the contrary that $D \neq \phi$

Then, as A is a well ordered set and

$D \subseteq A$, So D has first element.
let $d_0 \in D$ be the first element of D.
Then as $d_0 \in D$. So then by the definition of D, $f(d_0) < d_0$. As d_0 is first element of D and $f(d_0) < d_0$, so $f(d_0) \notin D$.
Next $f(d_0) < d_0 \Rightarrow f(f(d_0)) < f(d_0)$
$\Rightarrow f(d_0) \in D$.
which is a contradiction, so our supposition is wrong and hence $D = \phi$
$\Rightarrow$ for all $x \in A$, $x \lesssim f(x)$

2011
2012
2013

THEOREM:- If A and B are two similar well ordered sets then there is only one similarity mapping from A to B.

PROOF:-

Suppose on the contrary, There exist two similarity mappings $f: A \to B$ and $g: A \to B$ s.t $f \neq g$

Now as $g: A \to B$ is a similarity mapping
Then $g^{-1}: B \to A$ is also a similarity mapping.
Then $g^{-1} \circ f: A \to A$ is also a similarity mapp.
Now, As $f \neq g$ by assumption hypothesis, for some $x \in A$, $f(x) \neq g(x)$
As B is well ordered and $f(x), g(x) \in B$ s.t $f(x) \neq g(x) \Rightarrow \{f(x), g(x)\}$ has 1st element. Say $f(x)$ is the first element.
Then $f(x) < g(x)$
$\Rightarrow g^{-1}(f(x)) < g^{-1}(g(x))$
($\because g^{-1}$ is a similarity mapp)

$\Rightarrow (g^{-1} o f)(x) < x$
which is a contradiction, so our supposition is wrong and hence similarity mapping from $A \to B$ is unique.

2012

THEOREM:- A well ordered set can not be similar to any of its initial segment.

PROOF:-
Let A be a well ordered set and $a \in A$, and contrarily suppose $A \simeq S(a)$ then there exist a similar mapp $f: A \to S(a)$ As $a \in A \Rightarrow f(a) \in S(a) \Rightarrow f(a) < a$
which is a contradiction, so our supposition is wrong. Hence "A well ordered set can't be similar to any of its initial segment".

COMPARISON OF WELL ORDERED SET:-

Let A and B be the two well ordered sets, then either $A \sim B$ or $A \not\sim B$
if $A \not\sim B$ and A is similar to some initial segment of B, then A is said to be shorter than B and B is said to be longer than A.

THEOREM:- Let A be a well ordered set and $S \subseteq A$ with the property that if $a, b \in A$ and $a < b$ and $b \in S$, then $a \in S$. Then either $S = A$ or S is an initial segment of A.

PROOF:-

If $S = A$, then there is nothing to prove.
If $S \neq A$, then $S \subset A$ and $A \setminus S \neq \phi$
As A is well ordered set and $A \setminus S \neq \phi$ subset of A, so $A \setminus S$ has the 1st element.
Let $a_0 \in A \setminus S$ be the 1st element of $A \setminus S$.
As $a_0 \in A \setminus S \Rightarrow a_0 \notin S$.
Now consider, $S(a_0) = \{x \in A : x < a_0\}$
Now $x \in S(a_0) \Rightarrow x < a_0 \Rightarrow x \in S$ — P
$\Rightarrow S(a_0) \subseteq S$. — $*$
Now, let $x \in S$, as $a_0 \in A \setminus S \Rightarrow x \neq a_0$
Now as A is well ordered so is totally ordered.
$\Rightarrow$ either $x < a_0$ or $a_0 < x$
If $a_0 < x$, then $a_0 \in S$ by the given property.
Which is not possible because $a_0 \in A \setminus S$
$\Rightarrow a_0 \notin S$.
Hence $x < a_0 \Rightarrow x \in S(a_0)$
$\Rightarrow S \subseteq S(a_0)$ — $**$
from $*$ and $**$
$S = S(a_0)$

THEOREM:- Two different initial segment of a well ordered set cannot be similar.

PROOF:-

Let A be a well ordered set and $s(a)$, $s(b)$ be its two different initial segments i.e $a \neq b$. Since A is well ordered set and a well ordered set is totally ordered, then either $a < b$ or $b < a$.

Say $a < b$, Then $a \in S(b)$

As $S(b)$ is a well ordered set and $a \in S(b)$ so $S(a)$ is an initial segment of $S(b)$ and hence $S(a) \not\sim S(b)$

Because a well ordered set can't be similar to any of its initial segment"

Hence two different initial segment of a well ordered set can't be similar.

THEOREM:- Let A and B be two well ordered sets and let an initial segment $s(a)$ of A is similar to an initial segment of B. Then $s(a)$ is similar to a unique initial segment of B.

PROOF:-

Let us assume for $b, b' \in B$

$S(a) \sim S(b)$ and $S(a) \sim S(b')$

Then as relation of similarity is an equivalence relation so $S(b) \sim S(b')$.

Further as two different initial segments of a well ordered set can't be similar

and $S(b) \sim S(b')$ so $S(b) = S(b')$
Hence $S(a)$ is similar to a unique initial segment of B.

THEOREM:-
Let A and B be two well ordered sets such that an initial segment $S(a)$ of A is similar to an initial segment $S(b)$ of B. Then each initial segment of $S(a)$ is similar to an initial segment of $S(b)$ i.e $a' \leq a \Rightarrow S(a') \simeq S(b')$, where $b' \leq b$
Furthermore, if $f: S(a) \to S(b)$ is the similarity mapping to $S(b')$ and $S(b') = f(S(a'))$.

PROOF:-
As well as the 1st part is concern that is, each initial segment ~~of~~ $S(a')$ of $S(a)$ is similar to an initial segment $S(b')$ of $S(b)$, is obvious by the fact that every subset of a well ordered set is well ordered.
Now, given $f: S(a) \to S(b)$ is a similarity mapping To prove the restriction of f to $S(a')$ is also a similarity mapping.
let $f(a') = b'$ where $a' \in S(a)$
& $b' \in S(b)$

Now Let $a^* \in S(a') \Rightarrow a^* < a'$
$\Rightarrow f(a^*) < f(a')$
($\because$ f is similarity mapping)
$\Rightarrow f(a^*) < b' \quad \because f(a') = b'$
$\Rightarrow f(a^*) \in S(b')$
which shows that every element of $S(a')$ maps to an element of $S(b')$ under f, so f is restricted to $S(a')$ and is bijective and preserves the order.
Since f is onto, so $S(b') = f(S(a'))$

THEOREM:- Let A and B be two well ordered sets and let
$S = \{x : x \in A, S(x) \simeq S(y), \text{ where } y \in B\}$
Then either $S = A$ or S is an initial segment of A.

PROOF:-
Let $x \in S$ and $t < x$
Then by the definition of S, $S(x) \simeq S(y)$ for some $y \in B$
Now $t < x$ and $S(x) \simeq S(y)$
So $S(t)$ is similar to an initial segment of $S(y)$ and this initial segment is also an initial segment of B
$\Rightarrow S(t)$ is similar to an initial segment of B
$\Rightarrow t \in S$
$\Rightarrow S \subseteq A$ and has the property "If $x \in S$ and $t < x$, then $t \in S$", So Then

by one of the previous theorem, either $S = A$ or S is an initial segment of A

THEOREM:-

2013
2014

let A and B be two well ordered sets and $S = \{x : x \in A, S(x) \simeq S(y) \text{ for some } y \in B\}$

$T = \{y : y \in B, S(y) \simeq S(x) \text{ for some } x \in A\}$

Then $S \simeq T$.

PROOF:-

let $x \in S$, then by the definition of S. there is $y \in B$, s.t $S(x) \simeq S(y)$, so for each $x \in S$, there is a $y \in B$ which is of course unique, s.t $S(x) \simeq S(y)$

Define $f : S \to T$ by

$$f(x) = y \quad \text{s.t. } S(x) \simeq S(y)$$

The obviously f is bijective.

Now, It remains only to prove that f preserves the order also

let $x_1, x_2 \in S$ s.t $x_1 < x_2$

Now by the construction of S, there exist $y_1, y_2 \in B$ s.t $S(x_1) \simeq S(y_1)$ and $S(x_2) \simeq S(y_2)$

Now, $x_1 < x_2 \Rightarrow x_1 \in S(x_2)$

$\Rightarrow S(x_1)$ is an initial segment of $S(x_2)$

As $S(x_1) \simeq S(y_1)$, $S(x_2) \simeq S(y_2)$ and $S(x_1)$ is an initial segment of $S(x_2)$, So $S(y_1)$ is an initial segment of $S(y_2)$

$$\Rightarrow y_1 \in S(y_2)$$

$$\Rightarrow y_1 < y_2$$

Further $s(x_1) \sim s(y_1) \Rightarrow f(x_1) = y_1$
and $s(x_2) \sim s(y_2) \Rightarrow f(x_2) = y_2$

So $x_1 < x_2 \Rightarrow y_1 < y_2 \Rightarrow f(x_1) < f(x_2)$
$\Rightarrow f$ also preserves order
$\Rightarrow S \sim T$

THEOREM:-

If A and B are two well ordered sets, then

(i) $A \simeq B$ (ii) A is shorter than B
(iii) A is longer than B.

PROOF:-

Let us consider $S = \{x : x \in A, s(x) \simeq s(y)$, for som $y \in B\}$
and $T = \{y : y \in B, s(y) \sim s(x)$, for some $x \in A\}$

Then $S \sim T$, $S = A$ or S is an initial segment of A.
$T = B$ or T is an initial segment of B.
Now we have the following cases:
(i) If $S = A$ and $T = B$, then $S \sim T \Rightarrow A \simeq B$
(ii) If $S = A$ and T is an initial segment of B
Then $S \sim T \Rightarrow$ A is similar to an initial segment of B.
$\Rightarrow$ A is shorter than B.

(iii) If S is an initial segment of A and $T = B$, then $S \sim T \Rightarrow$ B is similar to an

initial segment of A.
$\Rightarrow$ B is shorter than A
$\Rightarrow$ A is longer than B.

(iv) If S is an initial segment $s(a)$ of A, $a \in A$ and T is an initial segment $S(b)$ of B, $b \in B$
Then $S \sim T \Rightarrow s(a) \sim s(b)$
$\Rightarrow a \in S$
But $S = s(a) \Rightarrow a \in s(a) \Rightarrow a < a$

$\Rightarrow$ A contradiction, so there are only there valid cases.

THEOREM:-

Let $\mathcal{A}$ be a family of initial segments of a well ordered set A. then there is an initial segment $s(a) \in \mathcal{A}$, which is shorter than every other segment in A.

PROOF:-

We know that $A \sim S(A)$, where $S(A)$ is the set of all initial segments of A. Since A is well ordered, so $S(A)$ is also well ordered.
$\Rightarrow \mathcal{A} \subseteq S(A)$ has a first element
Say $s(a)$ is the first element, then $s(a)$ is shorter than every other initial seg-

-ment in $\mathcal{A}$.

4 THEOREM:- Let $\mathcal{A}$ be any family of pairwise non similar well ordered sets, then there exist a set $A_0 \in \mathcal{A}$ s.t A_0 is shorter than every other set in $\mathcal{A}$.

PROOF:-

Let B be any set in $\mathcal{A}$ and define
$\beta = \{ X : X \in \mathcal{A}$ and X is shorter than $B \}$
if $\beta = \phi$, then B is the element in $\mathcal{A}$ which is shorter than every other element in $\mathcal{A}$.

If $\beta \neq \phi$, then if we show β has a shortest element i.e. set A_0, then considering the way β is defined, A_0 is the shortest element of $\mathcal{A}$.

Now by a well known theorem, every set $A \in \beta$ is similar to an initial segment $S(a)$ of B.
Let β' be the family of those initial segments of B, each of which is simi--lar to a set in β.

Then by a well know theorem β' contains an initial segment $S(a_0)$ which is shorter than every other initial seg--ment in β'. Then if $S(a_0) \simeq A_0$, then A_0 is shorter in β and Then A_0 is shorter in $\mathcal{A}$

ORDINAL NUMBERS:-

Definition:- Cardinal number of a well ordered set is called its ordinal number. So under the natural order ordinal numbers of $\phi, \{1\}, \{1,2\}, \{1,2,3\}$ are $0, 1, 2, 3$ respectively and are finite ordinals. All other ordinals are called transfinite numbers.

Ordinality of $\mathbb{N}$ is denoted by ω.
i.e $\text{ord}(\mathbb{N}) = \omega$.

Remark:-
If $A \sim B$, then $\text{ord}(A) = \text{ord}(B)$.

INEQUALITIES AND ORDINAL NUMBERS:-

Let A and B be two well ordered sets such that $\text{ord}(A) = \lambda$ and $\text{ord}(B) = \mu$. Then $\lambda < \mu$ if A is shorter than B i.e if A is similar to an initial segment of B.

THEOREM:-
If $\lambda = \text{ord}(A)$ and $\mu < \lambda$, then there exist a unique initial segment, say $S(a)$ of A such that $\mu = \text{ord}(S(a))$

PROOF:-
Suppose on the contrary that $S(a)$ and $S(b)$ are two initial segments of A s.t. $\mu = \text{ord}(S(a))$ and $\mu = \text{ord}(S(b))$

Then $ord(S(a)) = ord(S(b))$

$\Rightarrow S(a) \simeq S(b)$

which is a contradiction because two different initial segments of a well ordered set cannot be similar, Hence there exists a unique initial segment $S(a)$ of A. such that $\mu = ord(S(a))$.

THEOREM:-

Let λ be an ordinal number and $S(\lambda)$ be the set of all those ordinal numbers which are less than λ. Then $\lambda = ord(S(\lambda))$.

PROOF:-

Let $\lambda = ord(A)$ and let $S(A)$ denotes the set of all initial segments of A, then

$\phi : A \to S(A)$ defined by

$\phi(a) = S(a)$

Then obviously ϕ is bijective and further if $a, b \in A$ s.t $a < b$ Then $a \in S(b)$ and then $S(a)$ is an initial segment of $S(b)$ and also $S(a) \neq S(b) \Rightarrow S(a)$ is shorter than $S(b)$.

$\Rightarrow S(a) < S(b)$

$\Rightarrow \phi(a) < \phi(b)$

Converse is also obvious so ϕ is a similarity mapping.

$\Rightarrow A \sim S(A)$

Now, define $f : S(\lambda) \to S(A)$ by

$f(y) = S(a)$ if $y = \text{ord}(S(a))$

The existence of this function is guaranteed by the following fact:
for $y \in S(\lambda) \Rightarrow y < \lambda$ Then by the Theorem "If $\lambda = \text{ord}(A)$ and $y < \lambda$, then there exist a unique initial segment, say $S(a)$ of A s.t $y = \text{ord}(S(a))$"
There exists a unique initial segment say $S(a)$ of A s.t $y = \text{ord}(S(a))$

Now for each $S(a) \in S(A)$, $S(a) \not\simeq A$ because an initial segment cannot be similar to any of its initial segment
So then $S(a)$ is shorter than A.
Then $\text{ord}(S(a)) < \text{ord}(A)$

$$\Rightarrow \text{ord}(S(a)) < \lambda \Rightarrow \text{ord}(S(a)) = y$$
for some $y \in S(\lambda)$

and then $f(y) = S(a)$

$$\Rightarrow f \text{ is onto.}$$

Now, let $f(y_1) = f(y_2)$
where $y_1 = \text{ord}\, S(a_1)$
$y_2 = \text{ord}\, S(a_2)$

$$\Rightarrow S(a_1) = S(a_2)$$
$$\Rightarrow \text{ord}(S(a_1)) = \text{ord}(S(a_2))$$
$$\Rightarrow y_1 = y_2$$
$$\Rightarrow f \text{ is } 1\text{-}1$$

Now let $y_1, y_2 \in S(\lambda)$ s.t $y_1 < y_2$
and $y_1 = \text{ord}(S(a_1))$ & $y_2 = \text{ord}(S(a_2))$

Then $f(\mu_1) = S(a_1)$ and $f(\mu_2) = S(a_2)$
Now as $\mu_1 < \mu_2 \Rightarrow S(a_1)$ is shorter than $S(a_2)$.
$\Rightarrow S(a_1) < S(a_2) \Rightarrow f(\mu_1) < f(\mu_2)$
$\Rightarrow f$ preserves the order.
Hence f is a similarity mapping
$\Rightarrow S(\lambda) \simeq S(A)$.
Thus $A \simeq S(A)$ and $S(A) \simeq S(\lambda)$
$\Rightarrow A \simeq S(\lambda)$
$\Rightarrow ord(A) = ord(S(\lambda))$
$\Rightarrow \lambda = ord(S(\lambda))$.

ORDINAL ADDITION:-

Let A and B be two well ordered sets and are disjoint, with $\lambda = ord(A)$ and $\mu = ord(B)$ then $\lambda + \mu = ord(\{A; B\})$

REMARK:-

Show by an example that commutative law under addition does not hold for ordinal addition.

Sol:-

Let $A = \mathbb{N}$ and $B = \{a_1, a_2, a_3, \dots, a_n\}$ where A is well ordered under natural order and B is well ordered under natural order over its indices i.e. $a_i < a_j$ iff $i < j$ i.e $i \leq j$
Then $ord(A) = ord(\mathbb{N}) = \omega$ and $ord(B) = n$
Then $n + \omega = ord(\{B; A\})$

and $\omega + n = ord(\{A; B\})$

Now define $f : \{B; A\} \to A$ by

$$f(x) = i \quad \text{if } x = a_i$$
$$= n + i \quad \text{if } x = i$$

Then f is similarity mapp

$\Rightarrow \{B; A\} \sim A \Rightarrow ord(\{B; A\}) = ord(A)$

$\Rightarrow n + \omega = \omega \longrightarrow (1)$

Now $\{A; B\} = \{1, 2, 3, \ldots, a_1, a_2, a_3, \ldots, a_n\}$

Now $S(a_1) = \{1, 2, 3, \ldots\} \sim A$

$\Rightarrow A$ is similar to an initial segment of $\{A; B\}$

Then A is shorter than $\{A; B\}$

$\Rightarrow ord(A) < ord(\{A; B\})$

$\Rightarrow \omega < \omega + n \longrightarrow (2)$

from (1) & (2)

$n + \omega < \omega + n$

$\Rightarrow$ under ordinal addition commutative law does not hold.

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Question:-
Under Ordinal addition identity element is zero i.e for all ordinals λ, $0+\lambda = \lambda+0 = \lambda$

Sol: Let $o(A) = \lambda$. and $o(\phi) = 0$
Now $\{\phi ; A\} = A \Rightarrow \{\phi ; A\} \sim A$
$\Rightarrow ord(\{\phi ; A\}) = ord(A)$
$\Rightarrow 0 + \lambda = \lambda$

Also $\{A ; \phi\} = A \Rightarrow \{A ; \phi\} \sim A$
$\Rightarrow ord(\{A ; \phi\}) = ord(A)$
$\Rightarrow \lambda + 0 = \lambda$
$\Rightarrow 0$ is the additive identity in ordinal addition.

2012 Question:- For any ordinal λ, $\lambda+1$ is immediate successor of λ.
Sol:-
Let μ be the immediate successor of λ, then $S(\mu) = S(\lambda) \cup \{\lambda\}$
$= \{S(\lambda) ; \{\lambda\}\}$
$\Rightarrow ord(S(\mu)) = ord(S(\lambda)) + ord\{\lambda\}$
$\Rightarrow \mu = \lambda + 1$
$\Rightarrow \lambda+1$ is immediate successor of λ.

Multiplication of ORDINALS:-
Let A and B be two ordered sets, then $A \times B$ is also an ordered set under the order defined by $(a_1, b_1) < (a_2, b_2)$

iff $b_1 < b_2$ and if $b_1 = b_2$ then $a_1 < a_2$

This defined order is called Lexicographical order.

Remark:-

If A and B are well ordered then $A \times B$ is also well ordered under the Lexicographical order.

DEFINITION:-

If $ord(A) = \lambda$ and $ord(B) = \mu$, then $ord(A \times B) = \lambda\mu$

QUESTION:-

Show by an example that ordinal multiplication is not commutative.

Solution:-

Let $A = \mathbb{N}$ and $B = \{a, b\}$ be two well ordered sets

Then $ord(A) = ord(\mathbb{N}) = \omega$ and $ord(B) = 2$

Now $A \times B = \{(1,a), (2,a), (3,a), \ldots\ldots, (1,b), (2,b), \ldots\ldots\}$

Now $ord(A \times B) = \omega 2$

Now obviously $A \sim S(1, b)$

$\Rightarrow A = \mathbb{N}$ is similar to an initial segment of $A \times B$.

$\Rightarrow A = \mathbb{N}$ is shorter than $A \times B$

$\Rightarrow ord(A) < ord(A \times B)$

$\Rightarrow \omega < \omega 2$ ——— (i)

Now $B \times A = \{(a,1),(b,1),(a,2),(b,2),(a,3),(b,3),\dots\}$

Define $f: A \to B \times A$ by

$$f(n) = \begin{cases} (a, n/2) & \text{, if } n \text{ is even} \\ (b, \frac{n+1}{2}) & \text{, if } n \text{ is odd} \end{cases}$$

Then obviously f is a similarity mapp.

$\Rightarrow A \sim B \times A$

$\Rightarrow ord(A) = ord(B \times A)$

$\Rightarrow \omega = 2\omega$

$\Rightarrow 2\omega = \omega < \omega 2$

$\Rightarrow 2\omega < \omega 2$

$\Rightarrow$ multiplication is not commutative.

QUESTION:-

Show that 1 act as multiplicative identity in ordinal multiplication.

Sol:-

Let $ord(A) = \lambda$ and $B = \{b\}$

i.e $ord(B) = 1$

Now $A \times B = \{(a,b) : a \in A\}$

& $B \times A = \{(b,a) : a \in A\}$

Now, Define $\phi: A \to A \times B$ by

$\phi(a) = (a,b)$.

and $\psi: A \to B \times A$ by

$\psi(a) = (b,a)$

Then obviously ϕ and ψ are simil-arity mapping.

$\Rightarrow A \sim A \times B$ and $A \sim B \times A$

$\Rightarrow ord(A) = ord(A \times B)$ and $ord(A) = ord(B \times A)$

$\Rightarrow \lambda = \lambda.1$ and $\lambda = 1.\lambda$

$\Rightarrow 1.\lambda = \lambda.1 = \lambda$

$\Rightarrow$ 1 acts as multiplicative identity in ordinal multiplication

THEOREM:-

Prove that under Ordinal multiplication associative law holds.

PROOF:-

Let $ord(A) = \lambda$ and $ord(B) = \mu$
$ord(C) = \eta$

To prove $\lambda(\mu\eta) = (\lambda\mu)\eta$

For this we will prove

$A \times (B \times C) \sim (A \times B) \times C.$

For this, Define $f : A \times (B \times C) \to (A \times B) \times C$

by $f(a,(b,c)) = ((a,b),c)$

Then obviously f is bijective.

Now, To show f preserves order.

let $(a_1,(b_1,c_1)) < (a_2,(b_2,c_2))$

$\Rightarrow (b_1,c_1) < (b_2,c_2)$

and if $(b_1,c_1) = (b_2,c_2)$, then $a_1 < a_2$

If $(b_1,c_1) = (b_2,c_2)$ then $a_1 < a_2$

$\Rightarrow b_1 = b_2,\ c_1 = c_2$ and $a_1 < a_2$

then $a_1 < a_2$, $b_1 = b_2$ and $c_1 = c_2$

$\Rightarrow (a_1,b_1) < (a_2,b_2)$ and $c_1 = c_2$

$\Rightarrow ((a_1,b_1),c_1) < ((a_2,b_2),c_2)$

$\Rightarrow f(a_1,(b_1,c_1)) < f(a_2,(b_2,c_2))$

$\Rightarrow f$ preserves order.

Now, if $(b_1, c_1) < (b_2, c_2)$

$\Rightarrow c_1 < c_2$ and if $c_1 = c_2$ then $b_1 < b_2$.

If $c_1 < c_2$, then $((a_1, b_1), c_1) < ((a_2, b_2), c_2)$

$\Rightarrow f(a_1,(b_1,c_1)) < f(a_2,(b_2,c_2))$

$\Rightarrow f$ preserves order.

and if $c_1 = c_2$ and $b_1 < b_2$

$\Rightarrow (a_1, b_1) < (a_2, b_2)$ and $c_1 = c_2$

$\Rightarrow ((a_1, b_1), c_1) < ((a_2, b_2), c_2)$

$\Rightarrow f(a_1,(b_1,c_1)) < f(a_2,(b_2,c_2))$

$\Rightarrow f$ preserves order.

Hence for all possible cases f preserves order.

$\Rightarrow f$ is similarity mapping.

$\Rightarrow A \times (B \times C) \simeq (A \times B) \times C$

$\Rightarrow \operatorname{Ord}(A \times (B \times C)) = \operatorname{ord}((A \times B) \times C)$

$\Rightarrow \lambda(\mu\eta) = (\lambda\mu)\eta$

$\Rightarrow$ Under ordinal multiplication Associative law holds.

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CHOICE FUNCTION:-

let $\{A_i\}_{i\in I}$ be a non empty family of non empty subsets of X. Then A function $f: \{A_i\}_{i\in I} \to X$ defined by

$f(A_i) = a_i \in A_i$, for all $i \in I$

is called choice function.

CARTESIAN PRODUCT:-

let $\{A_i\}_{i\in I}$ be a non empty family of non empty sets. Then the cartesian product of this family is denoted by $\prod_{i\in I} A_i$ and is defined as the set of all choice functions on $\{A_i\}_{i\in I}$.

AXIOM OF CHOICE:-

Cartesian product of non empty family of non empty set is non empty

OR

There exist a choice function for any non empty family of non empty sets.

ZERMELO'S POSTULATE:-

let $\{A_i\}_{i\in I}$ be a non empty family of non empty disjoint sets, then there exist a subset B of $\bigcup_{i\in I} A_i$ s.t the intersection

of B and each A_i is non empty and contains exactly one element.

WELL ORDERING PRINCIPLE:-

Every set can be . well ordered.

THEOREM:- 2016 2015 2012

Show that axiom of choice is equivalent to Zermelo's postulate.

PROOF:-

Let $\{A_i\}_{i\in I}$ be a non empty family of non empty disjoint sets, and let f be a choice function on this family.
Now let $B = \{f(A_i) ; i\in I\}$, then $B\cap A_i$ for each $i\in I$ contains exactly one element $f(A_i)$ i.e $B\cap A_i = \{f(A_i)\}$ and this is because the family $\{A_i\}_{i\in I}$ has disjoint sets.
Conversely let $\{A_i\}_{i\in I}$ be a non empty family of non empty sets, which may not be disjoint.
Define $A_i^* = A_i \times \{i\}$, then the family $\{A_i^*\}_{i\in I}$ is a disjoint family because $A_i^* \cap A_j^* = \phi$ for all i,j s.t $i\neq j$ even if $A_i \cap A_j \neq \phi$.
So by Zermelo's postulate there exist a set $B \subseteq \bigcup_{i\in I} A_i^*$ s.t $B\cap A_i^*$ contains exactly one element say (a_i, i), where $a_i \in A_i$

Then, $f : \{A_i\}_{i \in I} \to \bigcup_{i \in I} A_i = X$ defined by $f(A_i) = a_i \in A_i$, is a Choice function.

Hence choice function and Zermelo's postulate are equivalent.

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SECTION II

MEASURE THEORY

CHAPTER NO 1.

BASIC CONCEPTS

Definition: Let X be a non empty set and $\mathcal{A}$ be a collection of subsets of X. Then $\mathcal{A}$ is said to be algebra on X if

i. For all $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

ii. For all $A \in \mathcal{A} \Rightarrow A' \in \mathcal{A}$

EXAMPLE:

Let $X = \{1, 2, 3, 4\}$ and $\mathcal{A}_1 = \{\phi, \{X\}, \{1\}, \{2,3,4\}\}$

and $\mathcal{A}_2 = \{\phi, X, \{1,2\}, \{3,4\}\}$

$\mathcal{A}_3 = \{\phi, X, \{1\}, \{3,4\}\}$

Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are algebra on X, but $\mathcal{A}_3$ is not an algebra on X.

REMARKS:

(i) if $\mathcal{A}$ is algebra on X and $A, B \in \mathcal{A}$

Then $A', B' \in \mathcal{A} \Rightarrow A' \cup B' \in \mathcal{A}$

$\Rightarrow (A' \cup B')' \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$

(ii) For any $X \neq \phi$, $\{\phi, X\}$ and $P(X)$ are algebra on X.

(iii) Let X be non empty set and $A \subseteq X$. Then $\{\phi, X, A, A'\}$ is an algebra on X.

2018 **Definition:-**
An algebra $\mathcal{A}$ on X is said to be σ-algebra if for every sequence $\{A_n\}$ of sets in $\mathcal{A} \Rightarrow \bigcup_n A_n \in \mathcal{A}$.
Then the pair $(X, \mathcal{A})$ is called measurable space and the elements in $\mathcal{A}$ are called measurable sets.

THEOREM:-
The intersection of any numbers of σ-algebras on X is again a σ-algebra on X.

PROOF:-
Let $\{\mathcal{A}_\alpha : \alpha \in I\}$ be a collection of σ-algebras on X. To prove $\bigcap_{\alpha\in I} \mathcal{A}_\alpha$ is a σ-algebra on X.

let $A, B \in \bigcap_{\alpha\in I} \mathcal{A}_\alpha \Rightarrow A, B \in \mathcal{A}_\alpha, \forall \alpha \in I$

$\Rightarrow A\cup B \in \mathcal{A}_\alpha, \forall \alpha\in I$, because for all $\alpha \in I$, $\mathcal{A}_\alpha$ is σ-algebra and hence algebra.

$\Rightarrow A\cup B \in \bigcap_{\alpha\in I} \mathcal{A}_\alpha$

let $A \in \bigcap_{\alpha\in I}\mathcal{A}_\alpha \Rightarrow A \in \mathcal{A}_\alpha, \forall \alpha\in I$.
$\Rightarrow A' \in \mathcal{A}_\alpha, \forall \alpha \in I$
$\Rightarrow A' \in \bigcap_{\alpha\in I}\mathcal{A}_\alpha$

$\Rightarrow \bigcap_{\alpha\in I}\mathcal{A}_\alpha$ is an algebra on X.

Now, let $\{A_n : n \in \mathbb{N}\}$ be a sequence of sets in $\bigcap_{\alpha \in I} \mathcal{A}_\alpha$

$\Rightarrow \{A_n : n \in \mathbb{N}\}$ is a sequence of sets in $\mathcal{A}_\alpha$, for all $\alpha \in I$

$\Rightarrow \bigcup_n A_n \in \mathcal{A}_\alpha, \forall \alpha \in I$. Because for all $\alpha \in I$ $\mathcal{A}_\alpha$ is a σ-algebra on X

$$\Rightarrow \bigcup_n A_n \in \bigcap_{\alpha \in I} \mathcal{A}_\alpha$$

$$\Rightarrow \bigcap_{\alpha \in I} \mathcal{A}_\alpha \text{ is a } \sigma\text{-algebra.}$$

REMARK:-

If $\mathcal{A}_1$ and $\mathcal{A}_2$ are two σ-algebras on X, then it is not necessary $\mathcal{A}_1 \cup \mathcal{A}_2$ is a σ-algebra on X.

e.g. Let $X = \{1, 2, 3, 4\}$

$\mathcal{A}_1 = \{\phi, X, \{1\}, \{2, 3, 4\}\}$, $\mathcal{A}_2 = \{\phi, X, \{2\}, \{1, 3, 4\}\}$

Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are σ-algebra on X.

Note that $\mathcal{A}_1 \cup \mathcal{A}_2 = \{\phi, X, \{1\}, \{2\}, \{1,3,4\}, \{2,3,4\}\}$

Then $\mathcal{A}_1 \cup \mathcal{A}_2$ is not a σ-algebra on X because $\{1\}, \{2\} \in \mathcal{A}_1 \cup \mathcal{A}_2$ but $\{1,2\} \notin \mathcal{A}_1 \cup \mathcal{A}_2$

$\Rightarrow \mathcal{A}_1 \cup \mathcal{A}_2$ is not an algebra on X and hence is not a σ-algebra.

2016 PROPOSITION:

let G be a family of subsets of X. Then there is a smallest σ-algebra on X containing G.

Proof:

let us consider

$\mathcal{H} = \{\mathcal{A} : \mathcal{A}$ is a σ-algebra on X and $G \subseteq \mathcal{A}\}$

Then $\mathcal{H}$ is non empty because at least $P(X)$ is a σ-algebra on X and $G \subset P(X)$

Next, if $\mathcal{H}_1 = \cap \mathcal{H} = \bigcap_{\mathcal{A} \in \mathcal{H}} \mathcal{A}$

Then $\mathcal{H}_1$ is a σ-algebra on X and $G \subseteq \mathcal{H}_1$

Now let $\mathcal{H}^*$ be another σ-algebra on X s.t $G \subseteq \mathcal{H}^*$, then by the construction of $\mathcal{H}$, $\mathcal{H}^* \in \mathcal{H}$

Then $\cap \mathcal{H} \subseteq \mathcal{H}^*$

$\Rightarrow \mathcal{H}_1 \subseteq \mathcal{H}^*$

$\Rightarrow \mathcal{H}_1$ is the smallest σ-algebra on X containing G.

Definition:

If G is a family of subsets of X. Then the smallest σ-algebra on X containing G is called σ-algebra on X generated by G.

PROPOSITION:

let $\{A_i\}$ be a sequence of sets in an algebra $\mathcal{A}$. Then there is a sequence $\{B_i\}$ of pairwise disjoint sets in $\mathcal{A}$ s.t. $\bigcup_n B_n = \bigcup_n A_n$

Proof:- Let $B_1 = A_1$

$B_2 = A_2 \setminus A_1$

$B_3 = A_3 \setminus (A_1 \cup A_2)$

$B_4 = A_4 \setminus (A_1 \cup A_2 \cup A_3)$

$\vdots$

$B_n = A_n \setminus (A_1 \cup A_2 \cup \dots \cup A_{n-1})$

$\vdots$

Then $B_n = A_n \setminus (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_{n-1})$

$= A_n \cap (A_1 \cup A_2 \cup A_3 \cup \dots \cup A_{n-1})'$

$\because A \setminus B = A \cap B'$

$= A_n \cap (A_1' \cap A_2' \cap A_3' \cap \dots \cap A_{n-1}') \in \mathcal{A}$

$\Rightarrow B_n \in \mathcal{A} \Rightarrow \{B_i\}$ is a sequence of sets in $\mathcal{A}$. Now we show the sequence $\{B_i\}$ has pairwise disjoint elements. For this let $m \neq n$ and consider B_m and B_n. To prove $B_m \cap B_n = \phi$

As $m \neq n$, so without any loss of generality say $m < n$

Now by the construction of B_m, $B_m \subseteq A_m$

$\Rightarrow B_m \cap B_n \subseteq A_m \cap B_n$

Now $A_m \cap B_n = A_m \cap (A_n \cap A_1' \cap A_2' \cap A_3' \cap \dots \cap A_{n-1}')$

$= A_m \cap A_n \cap A_1' \cap A_2' \cap \dots \cap A_{m-1}' \cap A_m' \cap A_{m+1}' \cap A_{m+2}' \cap \dots \cap A_{n-1}'$

$= A_m \cap A_m' \cap A_n \cap A_1' \cap A_2' \cap \dots \cap A_{m-1}' \cap A_{m+1}' \cap \dots \cap A_{n-1}'$

by commutative law

$= \phi \qquad \because A_m \cap A_m' = \phi$

$\Rightarrow B_m \cap B_n \subseteq A_m \cap B_n = \phi \Rightarrow B_m \cap B_n = \phi$

$\Rightarrow \{B_i\}$ has pairwise disjoint sets

Next, To prove $\bigcup_n B_n = \bigcup_n A_n$

As $B_n \subseteq A_n$

$\Rightarrow \bigcup_n B_n \subseteq \bigcup_n A_n \longrightarrow *$

Now let $x \in \bigcup_n A_n$

$\Rightarrow x \in A_n$, for some n

let m be the smallest natural number s.t $x \in A_m$ i.e $x \in A_m$ But $x \notin A_1, A_2, \ldots\ldots A_{m-1}$

$\Rightarrow x \in A_m$ and $x \notin A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_{m-1}$

$\Rightarrow x \in A_m \setminus (A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_{m-1})$

$\Rightarrow x \in B_m$

$\Rightarrow x \in \bigcup_n B_n$

$\Rightarrow \bigcup_n A_n \subseteq \bigcup_n B_n \longrightarrow **$

from $*$ and $**$

$\bigcup_n A_n = \bigcup_n B_n$

CHAPTER # 02

MEASURABLE SETS:-

Definition:-

Let $\mathcal{A}$ be a σ-algebra on X and μ be a real valued function on $\mathcal{A}$ and $\{A_i\}$ be a sequence of sets in $\mathcal{A}$, Then μ is said to be

(i) Finitely subadditive if

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mu(A_i)$$

(ii) Countablly subadditive

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

(iii) Finitely additive if

$$\mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

where $\{A_i\}_{i=1}^{n}$ pairwise disjoint sequence

(iv) Countablly additive or σ-additive if

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i), \text{ where } \{A_i\}$$

is pairwise disjoint sequence.

(v) montone if for $A, B \in \mathcal{A}$ and $A \subseteq B$

$$\mu(A) \leq \mu(B)$$

REMARK:- Obviously countablly additive (countablly subadditive) set function is finitely additive (finitely subadditive).

Definition:- let $\mathcal{A}$ be a σ-algebra on X and μ be an extended real valued function on $\mathcal{A}$. Then μ is said to be measure on $\mathcal{A}$, if

(i) $\mu(\phi)=0$ (ii) $\forall$, $A\in\mathcal{A} \Rightarrow \mu(A)\geq 0$

(iii) μ is countablly additive.

Definition:-

If μ is measure on $\mathcal{A}$, then the triplet $(X, \mathcal{A}, \mu)$ is called measure space.

EXAMPLE:-

let $X = \mathbb{N}$ and $\mathcal{A} = P(X) = 2^{\mathbb{N}}$

Define $\mu : 2^{\mathbb{N}} \to \overline{\mathbb{R}}$ by

$$\mu(A) = \begin{cases} \text{No. of elements in } A, & \text{if } A \text{ is finite} \\ \infty, & \text{if } A \text{ is infinite} \end{cases}$$

Then μ is measure and is called counting measure.

SOLUTION:-

(i) $\mu(\phi)=0$ $\because \phi$ is finite

(ii) let $A\subseteq \mathbb{N}$ i.e. $A\in 2^{\mathbb{N}}$, then if A is finite

Then $\mu(A)$ = No of elements in A

i.e $\mu(A)\geq 0$

If A is infinite Then $\mu(A)=\infty$.

So in either case $\mu(A)\geq 0$

(iii) let $\{A_n\}$ be a sequence of pairwise disjoint sets in 2^N, then there arises the following cases:

CASE-I:

If for all n, A_n is infinite
$\Rightarrow \bigcup_n A_n$ is also infinite.
Then $\mu(\bigcup_n A_n) = \infty$, Also for all n, as A_n is infinite so $\mu(A_n) = \infty$

$$\Rightarrow \sum_{n=1}^{\infty} \mu(A_n) = \infty \Rightarrow \mu(\bigcup_n A_n) = \sum_{n=1}^{\infty} \mu(A_n)$$

CASE II:

If for some n, A_n is infinite, Then for such n, $\mu(A_n) = \infty$ and then $\sum_{n=1}^{\infty} \mu(A_n) = \infty$
Also in this case $\bigcup_n A_n$ is infinite.
$\Rightarrow \mu(\bigcup_n A_n) = \infty$
$\Rightarrow \mu(\bigcup_n A_n) = \sum_{n=1}^{\infty} \mu(A_n)$

CASE III:

If for all n, A_n is finite, then this is obvious, due to the fact that $\{A_n\}$ is a sequence of pairwise disjoint sets
Then $\mu(\bigcup_n A_n) = \sum_{n=1}^{\infty} \mu(A_n)$

Hence in either case μ is countablly additive.

PROPOSITION:-

let (X, A, μ) be a measure space, then

(i)- if there exists $A \in \mathcal{A}$ s.t. $\mu(A) < \infty$
Then $\mu(\phi) = 0$

(ii)- μ is montone.

(iii)- If $\{A_n\}$ is a sequence of pairwise disjoint sets in $\mathcal{A}$, Then
$\mu(T) = \sum_{n=1}^{\infty} \mu(A_n \cap T) + \mu(T \cap (\bigcup_{n=1}^{\infty} A_n)')$
for any $T \in \mathcal{A}$.

PROOF:

(i) $\mu(A) = \mu(A \cup \phi)$
$= \mu(A) + \mu(\phi)$ $\because \mu$ is countablly additive
$\Rightarrow \mu(A) = \mu(A) + \mu(\phi)$ $\because \mu(A) < +\infty$
$\Rightarrow \mu(\phi) = 0$

(ii) let $A, B \in \mathcal{A}$ s.t. $A \subseteq B$
Now $B = (B \setminus A) \cup A$
and $(B \setminus A) \cap A = \phi$

Now $\mu(B) = \mu[(B \setminus A) \cup A]$
$= \mu(B \setminus A) + \mu(A)$
$\because \mu$ is countablly additive.

$\Rightarrow \mu(B) = \mu(B \setminus A) + \mu(A)$
$\Rightarrow \mu(B) \geq \mu(A)$ $\because \mu(B \setminus A) \geq 0$
$\Rightarrow \mu(B) \geq \mu(A) \Rightarrow \mu(A) \leq \mu(B)$
$\Rightarrow \mu$ is montone.

(iii) $T = T \cap X = T \cap [(\bigcup_{n=1}^{\infty} A_n) \cup (\bigcup_{n=1}^{\infty} A_n)']$

$\Rightarrow T = \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)\right] \cup \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)'\right]$

$\Rightarrow \mu(T) = \mu\left[\left\{\bigcup_{n=1}^{\infty} (T \cap A_n)\right\} \cup \left\{T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)'\right\}\right]$

$= \mu\left[\bigcup_{n=1}^{\infty} (T \cap A_n)\right] + \mu\left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)'\right]$

$\because \mu$ is additive.

$= \sum_{n=1}^{\infty} \mu(T \cap A_n) + \mu\left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)'\right]$

$\because \mu$ is countablly additive.

Definition:- A measure space (X, A, μ) is said to be finite if $\mu(X) < +\infty$. In general $A \in \mathcal{A}$ is said to be finite if $\mu(A) < +\infty$

Definition:- A measure space (X, A, μ) is called σ-finite if there is a sequence $\{E_n\}$ of sets in $\mathcal{A}$ with $X = \bigcup_n E_n$ and for all n, $\mu(E_n) < +\infty$

EXAMPLE:-

Counting measure on $\mathbb{N}$ is σ-finite

$\because \mathbb{N} = \bigcup_{n=1}^{\infty} \{n\}$

and $\mu(\{n\}) = 1 < +\infty$, for all n

Note that counting measure is not finite

$\because \mu(\mathbb{N}) = \infty$

EXAMPLE:-

let (X, A, μ) be a measure space, where

$$\mu_1(E) = \begin{cases} 0 & \text{if } E = \phi \\ \infty & \text{if } E \neq \phi \end{cases}$$

Then μ_1 is neither finite nor σ-finite

Example:

let $\mu_1(E) = \begin{cases} 0 & , \text{if } p \notin E \text{ , where } p \text{ is a fixed element of } X. \\ 1 & , \text{if } p \in E \end{cases}$

Then μ_1 is measure on X and it is both finite and σ-finite.

REMARK:

Every finite measure can be regarded as σ finite but converse is not true. e.g. counting measure is a σ-finite but is not finite

Definition:-

By an outer measure μ^*, we mean a non-negative extended real valued function on 2^X i.e. $\mu^* : 2^X \to \overline{\mathbb{R}}$ with the following properties.

(i) $\mu^*(\phi) = 0$

(ii) μ^* is montone

(iii) μ^* is countablly subadditive.

EXAMPLE: Define $\mu^* : 2^X \to [0, \infty]$ by

$$\mu^*(E) = \begin{cases} 0 & \text{if } E = \phi \\ 1 & \text{if } E \neq \phi \end{cases}$$

Then μ^* is an outer measure if X consists of more than one point, then μ^* is not countablly additive.

Solution:-

(i) By definition of μ^*, $\mu^*(\phi)=0$

(ii) μ^* IS MONTONE:-

let $A, B \in 2^X$ such that $A \subseteq B$

then we have the following cases:

CASE I: If $A=\phi$, $B=\phi$

$\Rightarrow \mu^*(A)=0$; $\mu^*(B)=0$ $\Rightarrow \mu^*(A)=\mu^*(B)$

CASE II: If $A=\phi$, $B\neq\phi$

$\Rightarrow \mu^*(A)=0$, $\mu^*(B)=1$ $\Rightarrow \mu^*(A) < \mu^*(B)$

CASE III: If $A\neq\phi$, $B\neq\phi$

$\Rightarrow \mu^*(A)=1$, $\mu^*(B)=1$ $\Rightarrow \mu^*(A)=\mu^*(B)$

Combining all three cases:

$$\mu^*(A) \leq \mu^*(B)$$

Hence $A\subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

$\Rightarrow \mu^*$ is montone.

(iii) let $\{E_n\}$ be any sequence of elements in 2^X, then

CASE I:

If for all n, $E_n=\phi$, Then $\mu^*(E_n)=0, \forall n$

$\Rightarrow \sum_{n=1}^{\infty} \mu^*(E_n)=0$

Then also

$\bigcup_{n=1}^{\infty} E_n = \phi \Rightarrow \mu^*(\bigcup_{n=1}^{\infty} E_n)=0$

$\Rightarrow \mu^* \bigcup_n E_n = \sum_{n=1}^{\infty} \mu^*(E_n)$

CASE II: If for all n, $E_n\neq\phi$

then $\mu^*(E_n) = 1$, $\forall n$

$\Rightarrow \sum_{n=1}^{\infty} \mu^*(E_n) = \infty$

Now also $\bigcup_n E_n \neq \phi \Rightarrow \mu^*(\bigcup_n E_n) = 1$

$\Rightarrow \mu^*(\bigcup_n E_n) < \sum_{n=1}^{\infty} \mu^*(E_n)$

CASE III: If for some n, $E_n \neq \phi$ then obviously $\mu^*(\bigcup_n E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$

Hence combining all the cases

$\mu^*(\bigcup_n E_n) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$

$\Rightarrow \mu^*$ is countablly subadditive.

Now, let $X = \{a, b\}$

$E_1 = \{a\}$, $E_2 = \{b\}$

$\Rightarrow E_1 \cup E_2 = \{a, b\}$

Now $\mu^*(E_1) = 1$, $\mu^*(E_2) = 1$, $\mu^*(E_1 \cup E_2) = 1$.

$\Rightarrow \mu^*(E_1 \cup E_2) < \mu^*(E_1) + \mu^*(E_2)$

even $E_1 \cap E_2 = \phi$

$\Rightarrow \mu^*$ is not additive.

LEBESGUE OUTER MEASURE:-

Lebesgue outer measure is a set function $m^*: 2^{\mathbb{R}} \to [0,\infty]$, defined by

$$m^*(A) = \inf\{\sum_n l(I_n) : A \subseteq \bigcup_n I_n\}$$

where infimum is taken over finite or countable sequences $\{I_n\}$ of open interval and $l(I_n)$ stands for length of open interval I_n.

REMARKS:-

(i) For every $A \in 2^{\mathbb{R}}$, $m^*(A) \geq 0$
$\because$ for any open interval I, $l(I) > 0$

(ii) $m^*(\phi) = 0$, clearly for every $\epsilon > 0$, however small $\phi \subseteq]-\frac{\epsilon}{4}, \frac{\epsilon}{4}[$ Then $m^*(\phi) \leq l(]-\frac{\epsilon}{4}, \frac{\epsilon}{4}[)$

$$\Rightarrow m^*(\phi) \leq \frac{\epsilon}{2}$$

$$\Rightarrow m^*(\phi) \leq \frac{\epsilon}{2} < \epsilon \Rightarrow m^*(\phi) < \epsilon$$

Since ϵ is arbitrary so $m^*(\phi) = 0$

(iii) let $x \in \mathbb{R}$ and $\{x\} \in 2^{\mathbb{R}}$, Then clearly for every $\epsilon > 0$, however small

$$\{x\} \subseteq]x - \frac{\epsilon}{4}, x + \frac{\epsilon}{4}[$$

$$\Rightarrow m^*(\{x\}) \leq l(]x - \frac{\epsilon}{4}, x + \frac{\epsilon}{4}[)$$

$$= x + \frac{\epsilon}{4} - x + \frac{\epsilon}{4} = \frac{\epsilon}{2} < \epsilon$$

$$\Rightarrow m^*\{x\} < \epsilon$$

Since ϵ is arbitrary, so $m^*(\{x\}) = 0$

(iv) let A, B be two subsets of $\mathbb{R}$ s.t. $A \subset B$, Then
$m^*(A) = \inf\{\sum_n l(I_n) : A \subseteq \bigcup_n I_n\}$, where I_n is a countable sequence of open intervals
and $m^*(B) = \inf\{\sum_n l(I_n') : B \subseteq \bigcup_n I_n'\}$, again $\{I_n'\}$ is a countable sequence of open intervals.
As $A \subseteq B$, so $\bigcup_n I_n \subseteq \bigcup_n I_n'$ in general.
$\Rightarrow$ In general $\sum_n l(I_n) \leq \sum_n l(I_n')$

$\Rightarrow \inf\{\sum_n l(I_n) : A \subseteq \bigcup_n A_n\} \leq \inf\{\sum_n l(I_n') : B \subseteq \bigcup_n I_n'\}$
$\Rightarrow m^*(A) \leq m^*(B)$
$\Rightarrow$ for $A \subseteq B \Rightarrow m^*(A) \leq m^*(B)$.
$\Rightarrow m^*$ is montone

PROPOSITION: v.v. imp

(i) If $\{A_n\}$ is any sequence of sets of real number, Then
$$m^*(\bigcup_n A_n) \leq \sum_n m^*(A_n)$$
i.e. m^* is countablly subadditive.

(ii) Lebesgue outer measure of a countable set is zero.

PROOF:-

If, for some n, $m^*(A_n)$ is infinite, then $\bigcup_n A_n$ is infinite, then
$$m^*(\bigcup_n A_n) = \sum_n m^*(A_n)$$ and both are infinite.

Same is the, if for all n, $m^*(A_n)$ is infinite.
Now assume for all n, $m^*(A_n)$ is finite
Now by the definition of $m^*(A_n)$
$m^*(A_n) = \inf \{ \sum_i l(I_{n,i}) : A_n \subseteq \bigcup_i I_{n,i} \}$, where

$I_{n,i}$ is a countable sequence of open intervals. Now by the definition of infimum for every $\epsilon > 0$, however small
$m^*(A_n) + \frac{\epsilon}{2^n} > \sum_i l(I_{n,i})$, for some countable

sequence $\{I_{n,i}\}$, such that $A_n \subseteq \bigcup_i I_{n,i}$

$$\Rightarrow \sum_i l(I_{n,i}) < m^*(A_n) + \frac{\epsilon}{2^n} \longrightarrow (1)$$

Now, as countable union of countable sets is countable so $\bigcup_n (\bigcup_i I_{n,i})$ is countable Union of $\bigcup_i I_{n,i}$ and $\bigcup_n A_n \subseteq \bigcup_n (\bigcup_i I_{n,i})$

Now by definition of m^*
$m^*(\bigcup_n A_n) \leq \sum_{n,i} l(I_{n,i})$

$$= \sum_n \left[\sum_i l(I_{n,i}) \right] < \sum_n \left[m^*(A_n) + \frac{\epsilon}{2^n} \right]$$

$$= \sum_n m^*(A_n) + \sum_n \epsilon/2^n$$

$$= \sum_n m^*(A_n) + \epsilon$$

$\Rightarrow m^*(\bigcup_n A_n) < \sum_n m^*(A_n) + \epsilon$
As ϵ is arbitrary, so

$$m^*(\bigcup_n A_n) \leq \sum_n m^*(A_n)$$

(ii) let $A = \{x_1, x_2, x_3, \ldots\}$ be a countable set. Then $A = \bigcup_n \{x_n\}$

$$m^*(A) = m^*(\bigcup_n \{x_n\}) \leq \sum_n m^*(\{x_n\})$$

$$= \sum_n (0) = 0$$

$\Rightarrow m^*(A) \leq 0$

$\Rightarrow m^*(A) = 0$

Obviously if A is finite, then also $m^*(A) = 0$

REMARK:-

From above proposition Lebesgue outer measure of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{N} \times \mathbb{N}$ etc are zero.

COROLLARY:-

Lebesgue outer measure is an outer measure because:

(i) $m^*(\phi) = 0$ (ii) m^* is motone

(iii) m^* is countablly subadditive.

v.imp 2010 2012

PROPOSITION:-

Lebesgue outer measure of an interval is its length.

Proof: We divide the proof in the following cases.

CASE I:

If interval is bounded and closed let $I = [a, b]$

Now, For any $\epsilon > 0$, however small

$[a, b] \subseteq]a-\epsilon, b+\epsilon[$

Then $m^*(I) = m^*([a,b])$

$\leq l(]a-\epsilon, b+\epsilon[)$

$= b-a+2\epsilon$

$\Rightarrow m^*(I) \leq b-a+2\epsilon$

$\Rightarrow m^*(I) \leq \text{Inf}\{b-a+2\epsilon : \epsilon > 0 \text{ and very very small}\}$

$= b-a$

$\Rightarrow m^*(I) \leq b-a \longrightarrow (i)$

Now, consider a countable sequence $\{I_m\}$ of open intervals s.t $I \subseteq \bigcup_m I_m$

As I is closed and bounded, so I is compact (By Heinborel theorem)

So then by the definition of compact, every open cover for I has a finite subcover. Thus as $\{I_m\}$ is an open cover so there is a finite open subcover $\{I_1, I_2, I_3, \ldots\ldots, I_n\}$ of I i.e $I \subseteq \bigcup_{i=1}^{n} I_i$

Now for $a \in I \Rightarrow a \in \bigcup_{i=1}^{n} I_i$, then $a \in I_{i^*}$ for some i^*

Let $I_{i^*} =]a_1, b_1[\Rightarrow a \in]a_1, b_1[$

$\Rightarrow a_1 < a < b_1$

Now if $b \notin]a_1, b_1[$, then $b_1 \leq b$

$\Rightarrow b_1 \in [a, b] \subseteq \bigcup_{i=1}^{n} I_i$

$\Rightarrow b_1 \in \bigcup_{i=1}^{n} I_i \Rightarrow b_1 \in I_{i'}$ for some i'

$\Rightarrow b_1 \in]a_2, b_2[$, where $]a_2, b_2[= I_{i'}$

$\Rightarrow a_2 < b_1 < b_2$

Now if $b \notin]a_2, b_2[$, then $b_2 \leq b$
further $a < b_2 \leq b$

$\Rightarrow b_2 \in [a,b] \subseteq \bigcup_{i=1}^{n} I_i$

$\Rightarrow b_2 \in \bigcup_{i=1}^{n} I_i \Rightarrow b_2 \in]a_3, b_3[$, where $]a_3, b_3[$ is one of $\{I_1, I_2, \dots\dots, I_n\}$

$\Rightarrow a_3 < b_2 < b_3$

Now continuing this process we get intervals $]a_1, b_1[,]a_2, b_2[,]a_3, b_3[\dots\dots]a_k, b_k[$ from $I_1, I_2, \dots\dots I_n$, $k \leq n$
s.t $a_i < b_{i-1} < b_i$

Since $\{I_i\}_{i=1}^{n}$ is finite, so this process must terminate at some stage $]a_k, b_k[$ and then $b \in]a_k, b_k[\Rightarrow b < b_k$

Now, note that $b_k > b$ and $a_1 < a$

$$\Rightarrow b_k - a_1 > b - a \longrightarrow *$$

Now $\sum_{i=1}^{n} l(I_i) \geq \sum_{i=1}^{k} l(I_i) \quad \because k \leq n$

$= (b_1 - a_1) + (b_2 - a_2) + \dots + (b_k - a_k)$

$= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \dots + (b_2 - a_2) + (b_1 - a_1)$

$= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2}) - \dots - (a_2 - b_1) - a_1$

$\geq b_k - a_1$

$> b - a$ by $*$

$\Rightarrow \sum_{i=1}^{n} l(I_i) > b - a$

Now $m^*([a,b]) = \inf\{\sum_n l(I_n) : [a,b] \subseteq \bigcup I_n\}$

$\geq b - a$

$$\Rightarrow m^*([a,b]) \geq b - a \longrightarrow \text{(ii)}$$

From (i) and (ii)

$m^*([a,b]) = b - a$

$= l([a,b])$

CASE II:

If I is an open bounded interval, then there exists, for every $\epsilon > 0$, a closed interval J s.t. $J \subseteq I$ and

$l(I) - \epsilon < l(J)$

$\Rightarrow l(I) - \epsilon < m^*(J)$ $\because m^*(J) = l(J)$ v/s J is closed interval (by case I)

Since ϵ is arbitrary

so $l(I) \leq m^*(J) \longrightarrow$ (i)

Now, $J \subseteq I$

and m^* is montone

so $m^*(J) \leq m^*(I)$

$\Rightarrow l(I) \leq m^*(J) \leq m^*(I)$ by (i)

$\Rightarrow l(I) \leq m^*(I) \longrightarrow$ (ii)

Now also $I \subset \bar{I}$

$\Rightarrow m^*(I) \leq m^*(\bar{I})$

$= l(\bar{I}) = l(I)$

$\Rightarrow m^*(I) \leq l(I) \longrightarrow$ (iii)

from (ii) and (iii)

$m^*(I) = l(I)$

CASE III:

If I is an infinite unbounded interval then for a given non-negative real number α, there is a closed interval J s.t. $J \subseteq I$ and $l(J) = \alpha$

and this result holds for how many la-
-rge α.
$\Rightarrow I \supseteq J$, and m^* is montone
so $m^*(I) \geq m^*(J) = l(J) = \alpha$
$\Rightarrow m^*(I) \geq \alpha$, for how many large α
$\Rightarrow m^*(I) = \infty = l(I)$
Hence combining all the above cases we conclude that Lebesgue outer me-
-asure of $[a,b]$, $]a,b[$, $]a,b]$, $[a,b[$
$[a,\infty[$, $]-\infty,b]$, $]-\infty,b[$, $]-\infty,\infty[$ is equal to length of corresponding int-
-erval
$\Rightarrow$ LOM of an interval is its length.

COROLLARY :-

Prove that $[a,b]$, $a \neq b$ is uncountable.

Proof:

Suppose on the contrary $[a,b]$ is countable then $m^*([a,b]) = 0$ which is a contra-
-diction, because the fact is
$m^*([a,b]) = l([a,b]) = b - a \neq 0$
$\because a \neq b$
$\Rightarrow$ our supposition is wrong and hence $[a,b]$ is uncountable.

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Definition:- Let $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, then translation of A by x is the set $x+A$, given by $x+A = \{x+a : a \in A\}$

Examples:

$2+[5,7[= [7,9[$

$3+]5,15[=]8,18[$

$3+]5,6[=]8,9[$

2019
2017
2014

PROPOSITION:-

Prove that LOM m^* is translation invarient.

PROOF:-

Let $E \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, Then translation of E by x means the set

$x+E = \{x+y : y \in E\}$

To prove $m^*(E) = m^*(x+E)$

Here arises two cases:

CASE I:-

If E is countable, then $x+E$ is also countable. Then as LOM of a countable set is zero so $m^*(E)=0$ & $m^*(x+E)=0$

$\Rightarrow m^*(E) = m^*(x+E)$

CASE II:-

If E is uncountable, then $x+E$ is also uncountable.

Now $m^*(E) = \text{Inf}\{\sum_n l(I_n) : E \subseteq \cup I_n\}$, where I_n is a countable sequence of open

intervals. Now for any $y \in E \Rightarrow y \in \bigcup_n I_n$
for some I_n
$\Rightarrow y \in I_n$, for some I_n
$\Rightarrow x+y \in x+I_n = J_n$
As I_n is an open interval so J_n is also an open interval.
Now for each $x+y \in x+E \Rightarrow x+y \in J_n$, for some n.

$\Rightarrow x+y \in x+E \Rightarrow x+y \in \bigcup_n J_n$

$\Rightarrow x+y \in x+E$
$\Rightarrow x+E \subseteq \bigcup_n J_n$
Next note that $l(I_n) = l(J_n)$; for all n
$\Rightarrow \sum_n l(I_n) = \sum_n l(J_n)$ such that for some $x+I_n = J_n$

$\Rightarrow \text{Inf}\{\sum_n l(I_n) : E \subseteq \bigcup_n I_n\} = \text{Inf}\{\sum_n l(J_n) : x+E \subseteq \bigcup_n J_n\}$
$\Rightarrow m^*(E) = m^*(x+E)$
$\Rightarrow m^*$ is translation invarient.

COROLLARY:-
Show that m^* is not one one.

Sol:
As for any $x \neq 0$ and $E \subseteq \mathbb{R}$
$x+E \neq E$
but by above theorem $m^*(x+E) = m^*(E)$
$\Rightarrow m^*$ is not 1 1

PROPOSITION:-
2017
If $m^*(A) = 0$, then for any $B \subseteq \mathbb{R}$
(i) $m^*(A \cup B) = m^*(B)$
(ii) If A is a set of rational numbers between 0 and 1, and $\{I_n\}$ is a countable sequence of open intervals s.t.

$A \subseteq \bigcup_n I_n$ then $\sum_n l(I_n) \geq 1$.

PROOF:-

(i) As m^* is countablly subadditive

$m^*(A \cup B) \leq m^*(A) + m^*(B)$

$= 0 + m^*(B) = m^*(B)$

$\Rightarrow m^*(A \cup B) \leq m^*(B) \longrightarrow *$

Now as $B \subseteq A \cup B$ and m^* is montone

so $m^*(B) \leq m^*(A \cup B) \longrightarrow **$

From $*$ and $**$

$m^*(A \cup B) = m^*(B)$.

(ii) As $\{I_n\}$ is a countable sequence of open intervals such that $A \subseteq \bigcup_n I_n$ and the smallest open interval containing all points of A is $]0,1[$ and so $A \subseteq]0,1[$, Thus $]0,1[\subseteq \bigcup_n I_n$

$\Rightarrow m^*(]0,1[) \leq m^*(\bigcup_n I_n) \leq \sum_n m^*(I_n)$

$\because m^*$ is montone and countablly subadditive

$\Rightarrow m^*(]0,1[) \leq \sum_n m^*(I_n) = \sum_n l(I_n)$

$\Rightarrow 1 \leq \sum_n l(I_n)$

$\Rightarrow \sum_n l(I_n) \geq 1$

v-imp 2016 2014 2011

PROPOSITION:

Given any set A and $\epsilon > 0$, there is an open set 'O' such that $A \subseteq O$ and $m^*(O) < m^*(A) + \epsilon$

Proof:

By definition $m^*(A) = \inf\{\sum_n l(I_n) : A \subseteq \bigcup_n I_n\}$

where I_n is a countable sequence of open intervals.

Now by definition of infimum for $\epsilon > 0$, there exists a countable sequence $\{I_n\}$ of open intervals such that $A \subseteq \bigcup_n I_n$ and

$m^*(A) + \epsilon > \sum_n l(I_n) \longrightarrow *$

Now put $O = \bigcup_n I_n$

As for each n, I_n is an open interval and an open interval is an open set and also as union of any number of open sets is an open set, So $\bigcup_n I_n$ is an open set

$\Rightarrow O$ is an open set and $A \subseteq O$

Next $m^*(O) = m^*(\bigcup_n I_n)$

$\leq \sum_n m^*(I_n)$ $\because m^*$ is countably subadditive

$= \sum_n l(I_n)$ $\because$ LOM of an interval is its length

$\Rightarrow m^*(O) \leq \sum_n l(I_n)$

$\Rightarrow \sum_n l(I_n) \geq m^*(O) \longrightarrow **$

From $*$ and $**$

$m^*(A) + \epsilon > \sum_n l(I_n) \geq m^*(O)$

$\Rightarrow m^*(A) + \epsilon > m^*(O)$

$\Rightarrow m^*(O) < m^*(A) + \epsilon$

DEFINITION:-

let $E \subseteq \mathbb{R}$, then E is said to be Lebesgue measurable set or simply measurable set if for each $A \subseteq \mathbb{R}$

$m^*(A) = m^*(A \cap E) + m^*(A \cap E')$

In this case A is called test set, as it is used to check the measurability of E.

REMARKS:

(i) $A = A \cap \mathbb{R} = A \cap (E \cup E')$
$= (A \cap E) \cup (A \cap E')$

$\Rightarrow m^*(A) = m^*[(A \cap E) \cup (A \cap E')]$
$\leq m^*(A \cap E) + m^*(A \cap E')$
$\because m^*$ is countablly and hence finitely subadditive.

$\Rightarrow$ $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E')$, and it always hold, so its means in order to prove E is measurable it is sufficient to prove

$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E')$

(ii) E is measurable iff E' is measurable
$\because m^*(A) = m^*(A \cap E) + m^*(A \cap E')$
iff $m^*(A) = m^*(A \cap (E')') + m^*(A \cap E')$

(iii) $m^*(A \cap \phi) + m^*(A \cap \phi')$
$= m^*(\phi) + m^*(A \cap \mathbb{R})$
$= 0 + m^*(A) = m^*(A)$

$\Rightarrow m^*(A) = m^*(A \cap \phi) + m^*(A \cap \phi')$
$\Rightarrow \phi$ is measurable
then by Remark (ii). $\phi' = \mathbb{R}$ is measurable

LEMMA:-

If $m^*(E) = 0$, then E is measurable set

Proof:

For any test set A, $A \cap E \subseteq E$
$\Rightarrow m^*(A \cap E) \leq m^*(E)$
$\because m^*$ is montone

$\Rightarrow m^*(A \cap E) \leq 0$

$\because m^*(E) = 0$ is given

but $m^*(A \cap E) \geq 0$ by definition of m^*

$\Rightarrow m^*(A \cap E) = 0$

Next $A \cap E' \subseteq A$

$\Rightarrow m^*(A \cap E') \leq m^*(A) \quad \because m^*$ is montone

$\Rightarrow m^*(A) \geq m^*(A \cap E') + m^*(A \cap E)$

$\because m^*(A \cap E) = 0$

$\Rightarrow E$ is measurable.

REMARK:

From above Lemma and by the fact that LOM of any singleton set, finite set countable set is zero. it concludes that all singleton sets, finite sets and countable sets are measurable sets.

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THEOREM:-

Union and intersection of two measurable sets is measurable set.

PROOF:

Let E_1 and E_2 be two measurable sets,

To prove $E_1 \cup E_2$ and $E_1 \cap E_2$ are measurable

First we prove $E_1 \cup E_2$ is measurable

let $A \subseteq \mathbb{R}$, Now using $A \cap E_1'$ as test set and the fact that E_2 is measurable, we have

$$m^*(A \cap E_1') = m^*(A \cap E_1' \cap E_2) + m^*(A \cap E_1' \cap E_2') \longrightarrow *$$

Now also we know that

$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_1' \cap E_2)$

$\Rightarrow m^*(A \cap (E_1 \cup E_2)) \leq m^*(A \cap E_1) + m^*(A \cap E_1' \cap E_2) \longrightarrow **$

$\because m^*$ is subadditive

Now,

$m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)'] = m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1' \cap E_2')]$

$\leq m^*(A \cap E_1) + m^*(A \cap E_1' \cap E_2) + m^*(A \cap E_1' \cap E_2')$ by $**$

$= m^*(A \cap E_1) + m^*(A \cap E_1')$ by $*$

$= m^*(A)$

$\because E_1$ is measurable

$\Rightarrow m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)'] \leq m^*(A)$

$\Rightarrow m^*(A) \geq m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)']$

$\Rightarrow E_1 \cup E_2$ is measurable

Now we show $E_1 \cap E_2$ is measurable

As E_1 and E_2 are measurable

$\Rightarrow E_1'$ and E_2' are measurable

$\Rightarrow E_1' \cup E_2'$ is measurable

$\Rightarrow (E_1' \cup E_2')'$ is measurable

$\Rightarrow E_1 \cap E_2$ is measurable

19-06-21

COROLLARY:-
Prove that union and intersection of finite number of measurable set is measurable.

PROOF:-
Let $\{E_1, E_2, \dots, E_n\}$ be a finite number of measurable sets.

To prove $\bigcup_{i=1}^{n} E_i$ is measurable.

As E_1 and E_2 are measurable

So $E_1 \cup E_2$ is measurable.

$\Rightarrow (E_1 \cup E_2) \cup E_3$ is measurable

$\Rightarrow E_1 \cup E_2 \cup E_3$ is measurable

$\Rightarrow (E_1 \cup E_2 \cup E_3) \cup E_4$ is measurable.

$\vdots$

$\Rightarrow E_1 \cup E_2 \cup \dots \cup E_n = \bigcup_{i=1}^{n} E_i$ is measurable

Now to prove $\bigcap_{i=1}^{n} E_i$ is measurable

Since E_i, for all i, is measurable

So E_i', for all i, is measurable.

Then by above $\bigcup_{i=1}^{n} E_i'$ is measurable.

$\Rightarrow \left[\bigcup_{i=1}^{n} E_i'\right]'$ is measurable.

$\Rightarrow \bigcap_{i=1}^{n} E_i$ is measurable.

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21-06-2021 LEMMA :-

Let A be any set of real numbers and $\{E_1, E_2, E_3, \ldots\ldots, E_n\}$ be a finite family of pairwise disjoint measurable sets, then

$$m^*\left[A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Proof:

We use principal of mathmatical induction to prove this result.

Obviously given is true for $n=1$.

Now suppose given is true for $n=k-1$

i.e

$$m^*\left[A \cap \left(\bigcup_{i=1}^{k-1} E_i\right)\right] = \sum_{i=1}^{k-1} m^*(A \cap E_i) \longrightarrow *$$

Now as E_i's are pairwise disjoint

So $A \cap \left(\bigcup_{i=1}^{k} E_i\right) \cap E_k = A \cap \left(\bigcup_{i=1}^{k} (E_i \cap E_k)\right) = A \cap E_k$

also

$$A \cap \left(\bigcup_{i=1}^{k} E_i\right) \cap E_k' = A \cap \left[\bigcup_{i=1}^{k} (E_i \cap E_k')\right]$$

Now as E_k is measurable, so if we use $A \cap \left(\bigcup_{i=1}^{k} E_i\right)$ as test set, then we have

$$m^*\left(A \cap \left(\bigcup_{i=1}^{k} E_i\right)\right) = m^*\left(A \cap \left(\bigcup_{i=1}^{k} E_i\right) \cap E_k\right) + m^*\left(A \cap \left(\bigcup_{i=1}^{k} E_i\right) \cap E_k'\right)$$

$$= m^*(A \cap E_k) + m^*\left(A \cap \left(\bigcup_{i=1}^{k-1} E_i\right)\right)$$

$$= m^*(A\cap E_k) + \sum_{i=1}^{k-1} m^*(A\cap E_i) \quad \text{by } *$$

$$\Rightarrow m^*\left[A\cap\left(\bigcup_{i=1}^{k} E_i\right)\right] = \sum_{i=1}^{k} m^*(A\cap E_i)$$

Hence given is true for $n=k$
So given is true for all finite k

$$\therefore m^*\left[A\cap\left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(A\cap E_i)$$

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THEOREM:-
let A be any set and $\{E_i\}$ be a sequence of pairwise disjoint measurable sets, then

$$m^*\left[A\cap\left(\bigcup_{i=1}^{\infty} E_i\right)\right] = \sum_{i=1}^{\infty} m^*(A\cap E_i)$$

PROOF:-
Since E_i's are pairwise disjoint
Therefore

$$A\cap\left(\bigcup_{i=1}^{n} E_i\right) \subseteq A\cap\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$\Rightarrow m^*\left[A\cap\left(\bigcup_{i=1}^{n} E_i\right)\right] \le m^*\left[A\cap\left(\bigcup_{i=1}^{\infty} E_i\right)\right]$$

$\because m^*$ is monotone

$$\Rightarrow \sum_{i=1}^{n} m^*(A\cap E_i) \le m^*\left[A\cap\left(\bigcup_{i=1}^{\infty} E_i\right)\right]$$

$$\because m^*\left[A\cap\left(\bigcup_{i=1}^{n} E_i\right)\right] = \sum_{i=1}^{n} m^*(A\cap E_i)$$

If $n \to \infty$, then

$$\Rightarrow \sum_{i=1}^{\infty} m^*(A\cap E_i) \le m^*\left[A\cap\left(\bigcup_{i=1}^{\infty} E_i\right)\right] \longrightarrow (1)$$

Also m^* is countablly subadditive $\rightarrow \boxed{\bigcup \subseteq \Sigma}$

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Therefore,

$$\sum_{i=1}^{\infty} m^*(A\cap E_i) \geq m^*(A\cap \bigcup_{i=1}^{\infty} E_i) \longrightarrow (2)$$

Comparing (1) and (2)

$$\Rightarrow \sum_{i=1}^{\infty} m^*(A\cap E_i) = m^*(A\cap (\bigcup_{i=1}^{\infty} E_i))$$

THEOREM:-

The class $\mathcal{M}$ of Lebesgue measurable set is a-algebra.

PROOF:-

We know that union of two measurable set is a measurable and also complement of a measurable set is measurable so $\mathcal{M}$ is an algebra.

Now let $\{E_i\}$ be a sequence of sets in $\mathcal{M}$. To prove $\bigcup_i E_i \in \mathcal{M}$. Let $E = \bigcup_i E_i$

Now, by a well known theorem, there is a pairwise disjoint sequence $\{F_i\}$ of measurable sets such that $\bigcup_i E_i = \bigcup_i F_i$

put $H_n = \bigcup_{i=1}^{n} F_i$, then for each n, H_n is measurable.

Next $H_n = \bigcup_{i=1}^{n} E_i \subset \bigcup_{i=1}^{\infty} E_i = E$

$$\Rightarrow H_n \subseteq E \quad \Rightarrow \quad E' \subseteq H_n'$$

$\Rightarrow A\cap E' \subseteq A\cap H_n'$ for every $A \subseteq \mathbb{R}$.

$\Rightarrow m^*(A\cap E') \leq m^*(A\cap H_n') \longrightarrow *$

$\because m^*$ is montone

Now as H_n is measurable, so

$$m^*(A) = m^*(A\cap H_n) + m^*(A\cap H_n')$$
$$\geq m^*(A\cap H_n) + m^*(A\cap E')$$
$$\because H_n = \bigcup_{i=1}^{n} F_i$$

$$= \sum_{i=1}^{n} m^*(A\cap F_i) + m^*(A\cap E')$$

$$\Rightarrow m^*(A) \geq \sum_{i=1}^{n} m^*(A\cap F_i) + m^*(A\cap E')$$

Since L.H.S is independent of n, so inequality holds for all n, so also holds when $n\to\infty$, thus:

$$m^*(A) \geq \sum_{i=1}^{\infty} m^*(A\cap F_i) + m^*(A\cap E')$$

$$= m^*\left[A\cap \left(\bigcup_{i=1}^{\infty} F_i\right)\right] + m^*(A\cap E')$$

$$= m^*\left[A\cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right] + m^*(A\cap E')$$
$$\because \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$$

$$= m^*(A\cap E) + m^*(A\cap E')$$

$\Rightarrow E$ is measurable

$\Rightarrow E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{M} \Rightarrow \mathcal{M}$ is a σ-algebra

REMARKS:

(i) It follows from above theorem and De-morgan's law that the σ-algebra $\mathcal{M}$ of measurable sets is closed under countable intersection.

ii) Now consider

$$m^*\left(A\cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) = \sum_{i=1}^{\infty} m^*(A\cap E_i)$$

where A is any set in $\mathbb{R}$ and E_i is

a sequence of pairwise disjoint measurable sets. Put $A = \mathbb{R}$, then $m^*(\mathbb{R} \cap \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(\mathbb{R} \cap E_i)$

$$\Rightarrow m^*(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(E_i)$$

$\Rightarrow m^*$ is countablly (and also finitely) additive in σ-algebra M.

DEFINITION:

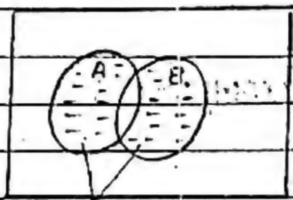
For any set A and B, their symmetric difference is denoted and defined by

$A \Delta B = (A \setminus B) \cup (B \setminus A)$

On can easily prove that

$A \Delta B = B \Delta A$ and

$(A \Delta B) \Delta C = A \Delta (B \Delta C)$



$(A \setminus B) \cup (B \setminus A)$

PROPOSITION:

If F is a measurable set and $m^*(F \Delta G)$ is zero i.e $m^*(F \Delta G) = 0$ then G is measurable.

PROOF:

$F \Delta G = (F \setminus G) \cup (G \setminus F)$

$\Rightarrow F \setminus G \subseteq F \Delta G$ and $G \setminus F \subseteq F \Delta G$

$\Rightarrow m^*(F \setminus G) \leq m^*(F \Delta G)$ and $m^*(G \setminus F) \leq m^*(F \Delta G)$

$\because m^*$ is montone

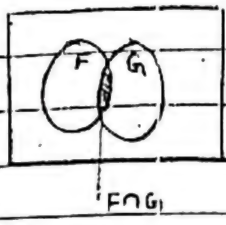
$\Rightarrow m^*(F \setminus G) \leq 0$ and $m^*(G \setminus F) \leq 0$

$\because m^*(F \Delta G) = 0$

$\Rightarrow F \setminus G$ and $G \setminus F$ both are measurable

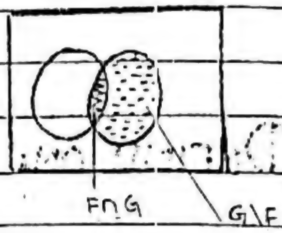
Next $F \cap G = F \cap (F \setminus G)'$

$\Rightarrow F \cap G$ is measurable



Next $G = (F \cap G) \cup (G \setminus F)$

$\Rightarrow G$ is measurable, because $F \cap G$ and $G \setminus F$ are measurable and union of two measurable sets is measurable.



Definition:-

24-06-2021 A set G is said to be G_δ set if it is the countable intersection of open sets.

EXAMPLE:

Any closed interval $[a,b]$, $a,b \in \mathbb{R}$ is a G_δ set, because $[a,b] = \bigcap_{n=1}^{\infty}]a-1/n, b+1/n[$ and open intervals are open sets in $\mathbb{R}$.

Definition:

A set F is said to be F_σ set if it is countable union of closed sets.

Example:

Any open interval $]a,b[$ is a F_σ set because $]a,b[= \bigcup_{n=1}^{\infty} [a+1/n, b-1/n]$

PROPOSITION:-

Given any set $A \subseteq \mathbb{R}$ and $\epsilon > 0$, there is a G_δ set G such that $A \subseteq G$ and $m^*(A) = m^*(G)$.

Proof:-

We know that for $A \subseteq \mathbb{R}$ and $\epsilon > 0$, there exists an open set O, such that $A \subseteq O$ and

$m^*(O) < m^*(A) + \epsilon$

If we choose $\epsilon = \frac{1}{n}$, then for each n, there exists open set O_n such that $A \subseteq O_n$

& $m^*(O_n) < m^*(A) + \frac{1}{n}$

$G = \bigcap_{n=1}^{\infty} O_n$, then being countable intersection G is G_δ set. Also as for all n, $A \subseteq O_n$

$\Rightarrow A \subseteq \bigcap_{n=1}^{\infty} O_n \Rightarrow A \subseteq G$

$\Rightarrow m^*(A) \leq m^*(G) \longrightarrow *$

Now $m^*(G) = m^*(\bigcap_{n=1}^{\infty} O_n) \leq m^*(O_n)$

$\because \bigcap_{n=1}^{\infty} O_n \subseteq O_n$ and m^* is montone

$< m^*(A) + \frac{1}{n}$

$\Rightarrow m^*(G) < m^*(A) + \frac{1}{n}$

when $n \to \infty$

$m^*(G) \leq m^*(A) \longrightarrow **$

from $*$ and $**$

$m^*(A) = m^*(G)$

THEOREM:-

For each $a \in \mathbb{R}$, the interval $]a, \infty[$ is measurable.

PROOF:-

Let A be any test set in $\mathbb{R}$, then To prove

$m^*(A) \geq m^*(A \cap]a,\infty[) + m^*(A \cap]a,\infty[')$

Let $A_1 = A \cap]a,\infty[$ and $A_2 = A \cap]a,\infty[' = A \cap]-\infty, a]$

i.e To prove $m^*(A) \geq m^*(A_1) + m^*(A_2)$

If $m^*(A) = \infty$ then inequality holds.

If $m^*(A) < \infty$ i.e. finite, then
$m^*(A) = \text{Inf}\{\sum_n l(I_n) : A \subseteq \bigcup_n I_n\}$, where I_n is a countable sequence of open intervals such that $A \subseteq \bigcup_n I_n$. Then further by the definition of infimum for every $\epsilon > 0$ there exist a countable sequence I_n of open intervals s.t. $A \subseteq \bigcup_n I_n$ and

$m^*(A) + \epsilon > \sum_n l(I_n)$

$\Rightarrow \sum_n l(I_n) < m^*(A) + \epsilon \longrightarrow *$

Let $J_n = I_n \cap]a, \infty[$ and $K_n = I_n \cap]-\infty, a]$

Then clearly $J_n \cap K_n = \phi$ and $J_n \cup K_n = I_n$

$$\Rightarrow l(I_n) = l(J_n) + l(K_n)$$
$$= m^*(J_n) + m^*(K_n)$$

Now $A_1 = A \cap]a, \infty[$.

Now, $A \subseteq \bigcup_n I_n \Rightarrow A \cap]a, \infty[\subseteq \bigcup_n I_n \cap]a, \infty[$
$$\Rightarrow A_1 \subseteq \bigcup_n (I_n \cap (]a, \infty[))$$
$$\Rightarrow A_1 \subseteq \bigcup_n J_n$$
$$\Rightarrow m^*(A_1) \leq m^*(\bigcup_n J_n)$$
$\because m^*$ is montone

$\Rightarrow m^*(A_1) \leq \sum_n m^*(J_n)$ $\because m^*$ is subadditive.

Similarly

$m^*(A_2) \leq \sum_n m^*(K_n)$

$\Rightarrow m^*(A_1) + m^*(A_2) \leq \sum_n [m^*(J_n) + m^*(K_n)]$

$= \sum_n l(I_n) < m^*(A) + \epsilon$

by *

Since ϵ is arbitrary so
$m^*(A_1) + m^*(A_2) \leq m^*(A)$
$\Rightarrow m^*(A) \geq m^*(A_1) + m^*(A_2)$
$\Rightarrow]a,\infty[$ is measurable.

LEMMA:

For any $a,b \in \mathbb{R}$, $]a,b[$ is measurable.

Proof:-

We know that $]a,\infty[$ is measurable
$\Rightarrow]a,\infty[' =]-\infty, a]$ is measurable, because complement of a measurable set is measurable. P
$\Rightarrow$ for each $n \in \mathbb{N}$, $]-\infty, b-\frac{1}{n}]$ is measurable
Then, as countable union of measurable set is measurable
so, $\bigcup_{n=1}^{\infty}]-\infty, b-\frac{1}{n}] =]-\infty, b[$ is measurable
Now,
$]a,b[=]a,\infty[\cap]-\infty,b[$
$\Rightarrow]a,b[$ is measurable.

THEOREM:-

Any open set O in $\mathbb{R}$ is measurable.

PROOF:-

We know that any open set can be expressed as union of open intervals and an open interval is a measurable set and also union of any number of measurable sets is measurable, so any open set

O in $\mathbb{R}$ is measurable.

COROLLARY:-
Any G_δ set is measurable.

Proof:-
let G_δ set is a countable intersection of open sets and an open set is measurable so it means G_δ set is countable intersection of measurable sets. Then as countable intersection of measurable sets is measurable so G_δ set is measurable

COROLLARY:-
For any $A \subseteq \mathbb{R}$, there exist a measurable set G such that $A \subseteq G$ and $m^*(A) = m^*(G)$

PROOF:-
We know that for any $A \subseteq \mathbb{R}$, there exist a G_δ set G such that $A \subseteq G$ and $m^*(A) = m^*(G)$. As G_δ set is measurable, so G is measurable with $A \subseteq G$ and $m^*(A) = m^*(G)$

COROLLARY:-
Any closed set in $\mathbb{R}$ is measurable.
Proof:-
let F be a closed set in $\mathbb{R}$ then F' is open in $\mathbb{R}$, then as in $\mathbb{R}$ open set is measurable, so F' is measurable, then

further as complement of a measurable set is measurable
so $(F')' = F$ is measurable.

COROLLARY:

Any F_σ set is measurable.

PROOF:-

F_σ set is countable union of closed sets. As closed sets are measurable and countable union of measurable sets is measurable so F_σ set is measurable.

REMARKS:

(i) The complement of F_σ set is G_δ set and conversely.

(ii) Since every open set is measurable, but converse is not true in general so it means metric topology $\mathcal{F}$ on $\mathbb{R}$ is a subfamily of $\mathcal{M}$.

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PROPOSITION:-

A subset E of $\mathbb{R}$ is measurable iff for given $\epsilon > 0$, there is an open set $O \supseteq E$ with $m^*(O \setminus E) < \epsilon$.

PROOF:-

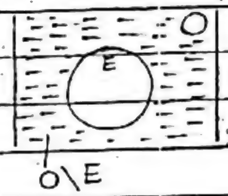
Let us assume E is measurable then we have the following cases:

CASE - I :

If $m^*(E) < \infty$ i.e. finite. Then by a well known theorem there is an open set O such that $E \subseteq O$ and $m^*(O) < m^*(E) + \epsilon$

$\Rightarrow m^*(O) - m^*(E) < \epsilon \quad \because m(E) < \infty$

Now $O = E \cup (O \setminus E)$



$\Rightarrow m^*(O) = m^*(E) + m^*(O \setminus E)$

E $\leftarrow$ $\because$ O and $O \setminus E$ are measurable and disjoint

$\Rightarrow m^*(O) - m^*(E) = m^*(O \setminus E)$

$\Rightarrow m^*(O \setminus E) = m^*(O) - m^*(E) < \epsilon$

$\Rightarrow m^*(O \setminus E) < \epsilon$

CASE II

If $m^*(E) = \infty$, Then we proceed as follows:

We know that

$\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$, where I_n is a bounded open interval for each n and for $m \neq n$, $I_m \cap I_n = \phi$

Put $E_n = E \cap I_n$.
Then for each n, E_n is measurable being the intersection of two measurable sets and also $m^*(E_n) < \infty$ i.e finite, because for each n, I_n is bounded.
Then by CASE - I for every $\epsilon > 0$ there exists an open set O_n s.t
$m^*(O_n \setminus E_n) < \frac{\epsilon}{2^n}$ and $O_n \supseteq E_n$

put $O = \bigcup_{n=1}^{\infty} O_n$, then O is open, because union of any number of open sets is open.

Now $\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E \cap I_n)$
$= E \cap (\bigcup_{n=1}^{\infty} I_n)$
$= E \cap \mathbb{R}$ $\because \mathbb{R} = \bigcup_{n=1}^{\infty} I_n$
$= E$

$\Rightarrow E = \bigcup_{n=1}^{\infty} E_n$

Now, for each n, $E_n \subseteq O_n \Rightarrow \bigcup_n E_n \subseteq \bigcup_n O_n$
$\Rightarrow E \subseteq O$

Now $(O \setminus E) = (\bigcup_n O_n) \setminus (\bigcup_n E_n) \subseteq \bigcup_n (O_n \setminus E_n)$
$\Rightarrow O \setminus E \subseteq \bigcup_n (O_n \setminus E_n)$
$\Rightarrow m^*(O \setminus E) \leq m^*[\bigcup_n (O_n \setminus E_n)] < \sum_n \epsilon/2^n$

$= \epsilon \sum_n 1/2^n$
$= \epsilon(1)$
$= \epsilon$

$\Rightarrow m^*(O \setminus E) < \epsilon$

Conversely, suppose that for $E \subseteq \mathbb{R}$, there

exists an open set $O \supseteq E$ with
$m^*(O \setminus E) < \epsilon$ for every $\epsilon > 0$
To prove E is measurable
Choose $\epsilon = \frac{1}{n}$, then by given condition, there exists an open set O_n
s.t. $O_n \supseteq E$ and $m^*(O_n \setminus E) < \frac{1}{n}$,
for all n

Put $G = \bigcap_n O_n$, the G is measurable, being the countable intersection of measurable sets.
($\because$ open set is measurable)
Further as $E \subseteq O_n$, for all n, so
$E \subseteq \bigcap_n O_n \Rightarrow E \subseteq G$

Next as $G = \bigcap_n O_n \Rightarrow G \subseteq O_n$, for all n
$\Rightarrow G \setminus E \subseteq O_n \setminus E$

$\Rightarrow m^*(G \setminus E) \le m^*(O_n \setminus E) < \frac{1}{n}$

$\Rightarrow m^*(G \setminus E) < \frac{1}{n}$, for all n

$\Rightarrow m^*(G \setminus E) = 0$

$\Rightarrow G \setminus E$ is measurable
As $E = G \cap (G \setminus E)'$
$\Rightarrow E$ is measurable.

PROPOSITION:

For a subset E of $\mathbb{R}$ and given $\epsilon>0$ there is an open set $O \supseteq E$ and $m^*(O \setminus E) < \epsilon$ iff there is a closed set $F \subseteq E$ and $m^*(E \setminus F) < \epsilon$.

PROOF:-

Let us assume there is an open set $O \supseteq E$ and $m^*(O \setminus E) < \epsilon$.

Since it holds for ever $E \subseteq \mathbb{R}$, so it also holds for $E' \subseteq \mathbb{R}$ i.e. for $\epsilon>0$ there is an open set $O \supseteq E'$ s.t $m^*(O \setminus E') < \epsilon$.

Put $F = O'$

Then as O is open, so O' i.e F is closed. Next $O \supseteq E' \Rightarrow O' \subseteq (E')'$

$$\Rightarrow O' \subseteq E \Rightarrow F \subseteq E$$

Next. $\epsilon > m^*(O \setminus E') = m^*(O \cap (E')')$

$\because A \setminus B = A \cap B'$

$$= m^*(O \cap E) = m^*(E \cap O)$$

$$= m^*(E \setminus O')$$

$$= m^*(E \setminus F)$$

$\Rightarrow m^*(E \setminus F) < \epsilon$

Conversely assume there is a closed set $F \subseteq E$ such that $m^*(E \setminus F) < \epsilon$

Since it holds for every $E \subseteq \mathbb{R}$, so in particular it holds for E' i.e. for $\epsilon>0$ there is a closed set $F \subseteq E'$ and $m^*(E' \setminus F) < \epsilon$

put $O = F'$, then as F is closed so F' is

open

As $F \subseteq E' \Rightarrow E \subseteq F' = O \Rightarrow E \subseteq O$

Now $m^*(E' \setminus F) < \epsilon$

$\Rightarrow m^*(E' \cap F') < \epsilon$

$\Rightarrow m^*(E' \cap O) < \epsilon \quad \Rightarrow m^*(O \cap E') < \epsilon$

$\Rightarrow m^*(O \setminus E) < \epsilon$

PROPOSITION:-

Let $E \subseteq \mathbb{R}$, then the following statements are equivalent.

(I) There is a G_δ set G with $E \subseteq G$, $m^*(G \setminus E) = 0$

(ii) There is a F_σ set F with $F \subseteq E$, $m^*(E \setminus F) = 0$

PROOF:-

(i) $\longrightarrow$ (ii)

Since the given statement is true for any $E \subseteq \mathbb{R}$ then it must hold for E'.

Therefore, for $E' \subseteq \mathbb{R}$ there is a G_δ set G containing E' such that

$m^*(G \setminus E') = 0$

$\Rightarrow m^*(G \cap E) = 0$

$\Rightarrow m^*(E \cap G) = 0$

$\Rightarrow m^*(E \setminus G') = 0$

Since complement of a G_δ set is F_σ set.

Therefore, $G' = F$, where F is F_σ set

and we have

$$m^*(E\setminus F)=0$$

(ii) $\longrightarrow$ (i)

For $E'\subseteq\mathbb{R}$ there is an F_σ set $F\subseteq E'$ such that $m^*(E'\setminus F)=0$

$\Rightarrow m^*(E'\cap F')=0$

$\Rightarrow m^*(F'\cap E')=0$

$\Rightarrow m^*(F'\setminus E)=0$

Since complement of F_σ set is G_δ set, therefore $F'=G$, where G is G_δ set and we have

$$m^*(G\setminus E)=0$$

PROPOSITION:-

Let $E\subseteq\mathbb{R}$, then the following statements are equivalent.

(i) E is measurable.

(ii) Given $\epsilon>0$, there is an open set $O\supseteq E$, with $m^*(O\setminus E)<\epsilon$

(iii) Given $\epsilon>0$ there is a closed set $F\subseteq E$ with $m^*(E\setminus F)<\epsilon$

(iv) There is a G_δ set $G\supseteq E$ with $m^*(G\setminus E)=0$

(v) There is an F_σ set $F\subseteq E$ with $m^*(E\setminus F)=0$

PROOF:-

(i) ⟹ (ii)

Already proved.

(ii) ⟹ (iii)

Already proved.

(iii) ⟹ (v) i.e given, for $\varepsilon > 0$, there is a closed set $F \subseteq E$ with $m^*(E \setminus F) < \varepsilon$. To prove there is a F_σ set F with $F \subseteq E$ - $m^*(E \setminus F) = 0$.

Take $\varepsilon = \frac{1}{n}$, then by given condition there exists F_n closed set, subset of E with

$m^*(E \setminus F_n) < \frac{1}{n}$

Let $F = \bigcup_{n=1}^{\infty} F_n$, then F is F_σ set & as for all $F_n \subseteq E$, so

$F = \bigcup_n F_n \subseteq E$ & $E \setminus F \subseteq E \setminus F_n$

$\Rightarrow m^*(E \setminus F) \le m^*(E \setminus F_n) < \frac{1}{n}$

When $n \to \infty$, $m^*(E \setminus F) = 0$.

(v) ⟹ (iv)

Already proved.

(iv) ⟹ (i) i.e. given there is a G_δ set G with $E \subseteq G$, $m^*(G \setminus E) = 0$ & to prove E is measurable.

~~As $G \setminus E$ is measurable, so~~

As $m^*(G \setminus E) = 0$, so $G \setminus E$ is measurable. Also then $(G \setminus E)'$ is measurable. Also G being G_δ set is measurable. Then

$G \cap (G \setminus E)' = E$ is measurable

Hence (i), (ii), (iii), (iv), (v) are equivalent.

Theorem:-

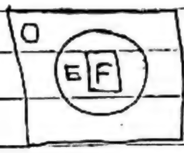
Prove that a set E is measurable iff for any $\varepsilon > 0$ there exists an open set $O \supseteq E$ and a closed set $F \subseteq E$ such that $m^*(O \setminus F) < \varepsilon$.

Proof:- Let us assume E is measurable & $\varepsilon > 0$, then

(i) there is an open set $O \supseteq E$ with $m^*(O \setminus E) < \varepsilon/2$ &

(ii) there is a closed set $F \subseteq E$ such that $m^*(E \setminus F) < \varepsilon/2$

Now



$$O \setminus F = (O \setminus E) \cup (E \setminus F)$$

$$\Rightarrow m^*(O \setminus F) \leq m^*(O \setminus E) + m^*(E \setminus F)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \Rightarrow m^*(O \setminus F) < \varepsilon.$$

Conversely assume for $E \subseteq \mathbb{R}$ & $\varepsilon > 0$ $\exists$ open set $O \supseteq E$ & closed set $F \subseteq E$ such that $m^*(O \setminus F) < \varepsilon$. To prove E is measurable.

Now $O \setminus E \subseteq O \setminus F$

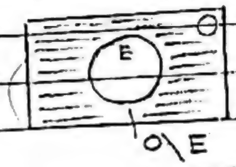
$$\Rightarrow m^*(O \setminus E) \leq m^*(O \setminus F) < \varepsilon$$

$$\Rightarrow m^*(O \setminus E) < \varepsilon$$

$$\Rightarrow E \text{ is measurable.}$$

THEOREM:-
Let $E \subseteq \mathbb{R}$ and $m^*(E) < \infty$, then E is measurable iff for $\epsilon > 0$, there is a finite union H of finite (i.e. bounded) open intervals s.t. $m^*(E \Delta H) < \epsilon$.

PROOF:-
Let us assume that E is measurable, then to prove for $\epsilon > 0$, there is a finite union H of finite open intervals such that $m^*(E \Delta H) < \epsilon$
Take $\epsilon > 0$, then by a well known theorem there is an open set $O \supseteq E$ such that $m^*(O \setminus E) < \epsilon/2 \longrightarrow *$
Now as $O = E \cup (O \setminus E)$ then by the finite additivity of m^*



$$m^*(O) = m^*(E) + m^*(O \setminus E)$$

Now as $m^*(E) < \infty$ i.e. finite and $m^*(O \setminus E) < \epsilon/2$ i.e. finite so $m^*(O)$ is finite.
Now as O, open set can be expressed as a countable union of pairwise disjoint open intervals I_n i.e.

$$O = \bigcup_{i=1}^{\infty} I_i$$

$$\Rightarrow m^*(O) = m^*\left(\bigcup_{i=1}^{\infty} I_i\right)$$
$$= \sum_{i=1}^{\infty} m^*(I_i) \quad \because m^* \text{ is } \sigma\text{-additive on } \mathcal{M}$$

As $m^*(O) < \infty$ so $\sum_{i=1}^{\infty} m^*(I_i) < \infty$
then by the convergence of the series,
there exists a +ve integer n, s.t.

$$\sum_{i=n+1}^{\infty} m^*(I_i) < \epsilon/2$$

Put $H = \bigcup_{i=1}^{n} I_i$

then,

$H \subseteq O$

and $O \setminus H = \bigcup_{i=n+1}^{\infty} I_i$

and thus $m^*(O \setminus H) = m^*(\bigcup_{i=n+1}^{\infty} I_i)$

$$= \sum_{i=n+1}^{\infty} m^*(I_i) < \epsilon/2$$

$$\Rightarrow m^*(O \setminus H) < \epsilon/2 \longrightarrow **$$

Next note that both E and H are subsets of O, and so

$$E \Delta H = (E \setminus H) \cup (H \setminus E)$$
$$\subseteq (O \setminus H) \cup (O \setminus E)$$

$\Rightarrow m^*(E \Delta H) \leq m^*(O \setminus H) + m^*(O \setminus E)$
$\because m^*$ is montone

$< \epsilon/2 + \epsilon/2 = \epsilon$
by $*$ and $**$

$\Rightarrow m^*(E \Delta H) < \epsilon$

Conversely, assume for $E \subseteq \mathbb{R}$ with $m^*(E) < \infty$ and $\epsilon > 0$, there is a finite union H of finite open

intervals such that $m^*(E \Delta H) < \epsilon/3$

To prove E is measurable.

Now, as $E \subseteq \mathbb{R}$ and $\epsilon > 0$, so then there exist an open set $O \supseteq E$ such that $m^*(O) < m^*(E) + \epsilon/3 \longrightarrow *$

let $H = \bigcup_{i=1}^{n} I_i$, where for each i, I_i is an open interval and put $J = O \cap H$, Then $J \subseteq O$ and $J \subseteq H$

Now,

Note that $E \Delta O \subseteq (E \Delta J) \cup (J \Delta O)$

$$\Rightarrow m^*(E \Delta O) < m^*(E \Delta J) + m^*(J \Delta O) \longrightarrow **$$

Next, note that

read $\leftarrow$ $E \setminus J = E \setminus H$

and as $J \subseteq H$ so $J \setminus E \subseteq H \setminus E$

Thus $E \Delta J = (E \setminus J) \cup (J \setminus E)$

$\subseteq (E \setminus H) \cup (H \setminus E)$

$= E \Delta H$

$$\Rightarrow E \Delta J \subseteq E \Delta H$$

$$\Rightarrow m^*(E \Delta J) \leq m^*(E \Delta H) < \epsilon/3$$

$$\Rightarrow m^*(E \Delta J) < \epsilon/3 \longrightarrow *'$$

Now, $E \subseteq J \cup (E \Delta J)$

$\Rightarrow m^*(E) \leq m^*(J) + m^*(E \Delta J)$

$< m^*(J) + \epsilon/3$

by $*'$

$\Rightarrow m^*(E) < m^*(J) + \epsilon/3 \longrightarrow *''$

As $m^*(O)$ is finite, and $J \subseteq O$

$\Rightarrow m^*(J)$ is finite, so then by $*''$

$m^*(E) - m^*(J) < \epsilon/3 \longrightarrow ***$

Now $m^*(O \Delta J) = m^*(O \setminus J) + m^*(J \setminus O)$

$= m^*(O \setminus J)$

$\because J \subseteq O,\ J \setminus O = \phi$

& $m^*(\phi) = 0$

$\Rightarrow m^*(O \Delta J) = m^*(O \setminus J)$

$= m^*(O) - m^*(J)$

$< m^*(E) - m^*(J) + \epsilon/3$

by $*$

$< \epsilon/3 + \epsilon/3$

by $***$

$\Rightarrow m^*(O \Delta J) < 2\epsilon/3 \longrightarrow (**)'$

Hence $E \Delta O = (E \setminus O) \cup (O \setminus E)$

$= \phi \cup (O \setminus E) = O \setminus E$

$\Rightarrow m^*(O \setminus E) = m^*(E \Delta O)$

$< m^*(E \Delta J) + m^*(J \Delta O)$

by $**$

$< \epsilon/3 + 2\epsilon/3$ by $*'$ and $(**)'$

$= \epsilon$

$\Rightarrow m^*(O \setminus E) < \epsilon$

$\Rightarrow E$ is measurable.

Definition:-

Let E be a measurable set, then Lebesgue measure of E is defined to be Lebesgue outer measure of E and is denoted by $m(E)$

THEOREM:

Let $\{E_i\}$ be a sequence of measurable sets Then

(i) m is countablly subadditive

(ii) m is finitely additive provided $\{E_i\}$ are pairwise disjoint.

(iii) m is countablly additive provided $\{E_i\}$ are pairwise disjoint

(iv) m is montone.

(v) m is translation invariant.

PROOF:-

(I) As $\{E_i\}$ is a sequence of measurable sets, so $\bigcup_i E_i$ is also measurable. Then

$$m(\bigcup_i E_i) = m^*(\bigcup_i E_i)$$
$$\leq \sum_{i=1} m^*(E_i)$$

$$= \sum_i m(E_i) \Rightarrow m(\bigcup_i E_i) \leq \sum_i m(E_i)$$

$\Rightarrow m$ is countablly subadditive.

(ii) We know that for any $A \subseteq \mathbb{R}$

$$m^*(A \cap (\bigcup_{i=1}^{n} E_i)) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

Put $A = \mathbb{R}$, Then

$$m^*(\mathbb{R} \cap (\bigcup_{i=1}^{n} E_i)) = \sum_{i=1}^{n} m^*(\mathbb{R} \cap E_i)$$

$$\Rightarrow m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(E_i)$$

$$\Rightarrow m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(E_i)$$

$\Rightarrow m$ is finitely additive.

(iii) We know that for any $A \subseteq \mathbb{R}$;

$$m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

Putting $A = \mathbb{R}$ and proceeding as in (ii) we obtain

$$m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$$

$\Rightarrow m$ is countablly additive.

(iv)

Let A and B be two measurable sets such that $A \subseteq B$. Then $B \setminus A$ is also measurable and $B = A \cup (B \setminus A)$. Then by additivity of m

$$m(B) = m(A) + m(B \setminus A)$$
$$\geq m(A)$$
$$\because m(B \setminus A) = m^*(B \setminus A) \geq 0$$

$\Rightarrow m(A) \leq m(B)$

$\Rightarrow m$ is montone.

(V) Let E be measurable and $y \in \mathbb{R}$, First we show that $E + y$ is also measurable. Since E is measurable so for $\epsilon > 0$, there exist an open set $O \supseteq E$, such that $m^*(O \setminus E) < \epsilon$ and convessely. Now $O \supseteq E \Rightarrow O + y \supseteq E + y$. As O

is an open set. Then $O+y$ is also an open set. Also $(O+y)\setminus(E+y) = (O\setminus E)+y$

Then $m^*((O+y)\setminus(E+y)) = m^*((O\setminus E)+y)$
$= m^*(O\setminus E) < \epsilon$

$\because m^*$ is translation invariant.

$\Rightarrow m^*((O+y)\setminus(E+y)) < \epsilon$

$\Rightarrow$ We have found an open set $O+y \supseteq E+y$ such that $m^*((O+y)\setminus(E+y)) < \epsilon$ for $\epsilon > 0$. Hence $E+y$ is measurable.

Thus

$m(E+y) = m^*(E+y)$ $\because E+y$ is measurable

$= m^*(E)$

$\Rightarrow m(E+y) = m(E)$

$\Rightarrow m$ is translation invariant.

Definition:

A measure space $(X, \mathcal{A}, \mu)$ (measure) is said to be complete if each subset of a set of measure zero is itself measurable i.e. if $A \in \mathcal{A}$ & $\mu(A) = 0$ then for $B \subseteq A \Rightarrow B \in \mathcal{A}$.

EXAMPLES:-

(i)

$(\mathbb{R}, \mathcal{M}, m)$ is complete. Because if $A \in \mathcal{M}$ such that $m(A) = 0$ and $B \subseteq A$

Then as $m^*(A) = m(A) = 0$

$\Rightarrow m^*(B) \leq 0 \Rightarrow m^*(B) = 0$

$\Rightarrow B$ is measurable $\Rightarrow B \in \mathcal{M}$

(ii)

Counting measure space $(N, \mathcal{A}, \mu)$ is complete. Because e.g.

if $A \in \mathcal{A}$ s.t. $\mu(A) = 0 \Rightarrow A$ is empty

Then for any $B \subseteq A \Rightarrow B$ is empty

$\Rightarrow B \in \mathcal{A} \Rightarrow$ Result is obvious.

$(X, \mathcal{A}, \mu)$
$\mathcal{A} = \{\phi, X\}$
$X = \{a, b\}$
$X \in \mathcal{A}$
$\mu(X) = 0$
$A \subset X$
$\mu(A) = 0$
But $A \notin \mathcal{A}$

(iii) Let X be any set with more than one elements.

$\mathcal{A} = \{\phi, X\}$ and μ is a zero measure on X. Then $(X, \mathcal{A}, \mu)$ is not complete

$\because \mu(X) = 0$

and $A \subset X$ $\because X$ has more than one element

Then $\mu(A) = 0$ but $A \notin \mathcal{A}$.

2012

THEOREM:-

Let $\{E_n\}$ be a sequence of measurable sets

(1) If $\{E_n\}$ is a decreasing sequence and $m(E_1) < \infty$ then $m(\bigcap_n E_n) = \lim_{n \to \infty} m(E_n)$

Show by an example that the condition $m(E_1) < \infty$ is necessary for the conclusion.

(2) If $\{E_n\}$ is an increasing sequence then $m(\bigcup_n E_n) = \lim_{n \to \infty} m(E_n)$

PROOF:-

Put $E = \bigcap_n E_n$ and $F_i = E_i \setminus E_{i+1}$ for all i, then

$A \setminus B$
$A \cap B'$

(i) $F_i = E_i \setminus E_{i+1}$
$= E_i \cap E'_{i+1}$

$\Rightarrow$ for each i, F_i is measurable

$\Rightarrow$ $\{F_i\}$ is a sequence of measurable sets.

(ii) For any $i \neq j$, say $i > j$, then $i \geq j+1$
Then $F_i \cap F_j = (E_i \cap E'_{i+1}) \cap (E_j \cap E'_{j+1})$

$= (E_i \cap E'_{j+1}) \cap (E'_{i+1} \cap E_j)$
$= \phi \cap (E'_{i+1} \cap E_j)$

$= \phi$

$\Rightarrow$ for $i \neq j$, $F_i \cap F_j = \phi$
$\Rightarrow$ $\{F_i\}$ is a measurable sequence of pairwise disjoint sets.

(iii) We now prove that
$E_1 \setminus E = \bigcup_{i=1}^{\infty} F_i$, where $E = \bigcap_{i=1}^{\infty} E_i$

let $x \in E_1 \setminus E$
$\Rightarrow x \in E_1$ and $x \notin E$
As $x \notin E \Rightarrow x \notin \bigcap_{i=1}^{\infty} E_i$, As $\{E_i\}$ is a decreasing sequence, so there is some E_j such that $x \in E_j$ but $x \notin E_{j+1}$
$\Rightarrow x \in E_j \setminus E_{j+1}$
$\Rightarrow x \in F_j \Rightarrow x \in \bigcup_{i=1}^{\infty} F_i$

$\Rightarrow E_1 \setminus E \subseteq \bigcup_{i=1}^{\infty} F_i \longrightarrow *$

Now let $x \in \bigcup_{i=1}^{\infty} F_i$

$\Rightarrow x \in F_i$, for some i

$\Rightarrow x \in E_i \setminus E_{i+1}$

$\Rightarrow x \in E_i$ and $x \notin E_{i+1}$

As $x \in E_i$, for some i and $\{E_i\}$ is a decreasing sequence so $x \in E_1$

As $x \notin E_{i+1}$, so $x \notin \bigcap_{i=1}^{\infty} E_i = E$

$\Rightarrow x \notin E$

$\Rightarrow x \in E_1$ and $x \notin E$

$\Rightarrow x \in E_1 \setminus E$

$\Rightarrow \bigcup_{i=1}^{\infty} F_i \subseteq E_1 \setminus E \longrightarrow **$

From $*$ and $**$

$\bigcup_{i=1}^{\infty} F_i = E_1 \setminus E \longrightarrow *'$

and also note that both sides are measurable Now as

$E = \bigcap_{i=1}^{\infty} E_i \Rightarrow E \subseteq E_i$, for all i.

$\Rightarrow E \subseteq E_1$

$\Rightarrow m(E) \leq m(E_1)$

$\because m$ is montone

$\Rightarrow m(E)$ is finite because $m(E_1)$ is finite

Also $E \subseteq E_1 \Rightarrow E_1 = E \cup (E_1 \setminus E)$

$\Rightarrow m(E_1) = m(E) + m(E_1 \setminus E)$

$\because m$ is finitely odditive.

$\Rightarrow m(E_1) - m(E) = m(E_1 \setminus E) \longrightarrow *''$

$\because m^*(E)$ is finite.

Similarly by the fact $E_{i+1} \subseteq E_i$, so by $*''$

$$m(E_i \setminus E_{i+1}) = m(E_i) - m(E_{i+1}).$$

Now by $*'$

$$m(E_1 \setminus E) = m\left(\bigcup_{i=1}^{\infty} F_i\right)$$

$$= \sum_{i=1}^{\infty} m(F_i) \quad \because m \text{ is a additive}$$

$$m(E_1 \setminus E) = \sum_{i=1}^{\infty} m(E_i \setminus E_{i+1})$$

$$\because F_i = E_i \setminus E_{i+1}$$

$$\Rightarrow m(E_1) - m(E) = \sum_{i=1}^{\infty} (m(E_i) - m(E_{i+1})) \quad \text{by } *''$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} [m(E_i) - m(E_{i+1})]$$

$$= \lim_{n\to\infty} [m(E_1) - m(E_2) + m(E_2) - m(E_3) + m(E_3) - m(E_4) + \cdots + m(E_n) - m(E_{n+1})]$$

$$= \lim_{n\to\infty} [m(E_1) - m(E_{n+1})]$$

$$m(E_1) - m(E) = m(E_1) - \lim_{n\to\infty} m(E_{n+1})$$

$$\because m(E_1) < \infty$$

$$\Rightarrow m(E) = \lim_{n\to\infty} m(E_{n+1})$$

$$= \lim_{n\to\infty} m(E_n)$$

$$\Rightarrow m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n)$$

COUNTER EXAMPLE:-

Let $E_n =]n, \infty[$, then $\{E_n\}$ is a sequence of measurable sets and is a decreasing sequence

Here $m(E_1) = m(]1, \infty[) = m^*(]1, \infty[) = \infty$

$\Rightarrow m(E_1) = \infty$

$\Rightarrow m(E_1) \not< \infty$

Then $\bigcap_{n=1}^{\infty} E_n = \phi \Rightarrow m(\bigcap_{n=1}^{\infty} E_n) = m(\phi) = 0$

Now $\lim_{n\to\infty} m(E_n) = \infty$

$\Rightarrow m(\bigcap_{n=1}^{\infty} E_n) \neq \lim_{n\to\infty} m(E_n)$

(2) Given sequence $\{E_n\}$ is increasing

To show $m(\bigcup_n E_n) = \lim_{n\to\infty} m(E_n)$

put $B_1 = E_1$

$B_2 = E_2 \setminus E_1$

$B_3 = E_3 \setminus E_2$

$\vdots$

$B_n = E_n \setminus E_{n-1}$

Then for each n, B_n is measurable

$\because B_n = E_n \cap E_{n-1}'$ and for $m \neq n$.

$B_n \cap B_m = \phi$

$\Rightarrow \{B_i\}$ is a sequence of measurable sets with pairwise disjoint element and

$E_n = \bigcup_{i=1}^{n} B_i$

$\Rightarrow \bigcup_n E_n = \bigcup_{i=1}^{\infty} B_i \Rightarrow m(\bigcup_n E_n) = m(\bigcup_{i=1}^{\infty} B_i)$

$= \sum_{i=1}^{\infty} m(B_i)$

$= \lim_{n\to\infty} \sum_{i=1}^{n} m(B_i)$

$= \lim_{n\to\infty} m(\bigcup_{i=1}^{n} B_i)$

$$= \lim_{n\to\infty} m(E_n)$$

$$\Rightarrow m(\bigcup_n E_n) = \lim_{n\to\infty} m(E_n)$$

THEOREM:-

If E_1, E_2 are two measurable sets, then
$m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$

PROOF:-

Since E_1 is measurable, so for any set $A \subseteq \mathbb{R}$

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1')$$

In particular for $E_2 \subseteq \mathbb{R}$

$$m^*(E_2) = m^*(E_2 \cap E_1) + m^*(E_2 \cap E_1')$$

$$\Rightarrow m^*(E_2) = m^*(E_1 \cap E_2) + m^*(E_1' \cap E_2) \longrightarrow (i)$$

(∵ intersection is commutative)

As E_2 is also measurable, so in the same way

$$m^*(E_1) = m^*(E_1 \cap E_2) + m^*(E_1 \cap E_2') \longrightarrow (ii)$$

Since E_1 and E_2 are measurable so we write m in place of m^* in (i) and (ii) and then add, we get

$$m(E_1) + m(E_2) = m(E_1 \cap E_2) + m(E_1 \cap E_2') + m(E_1 \cap E_2) + m(E_1' \cap E_2) \longrightarrow *$$

As

$(E_1 \cup E_2) = (E_1 \cap E_2') \cup (E_2 \cap E_1') \cup (E_1 \cap E_2)$ +

Since $E_1 \cap E_2'$, $E_2 \cap E_1'$ and $E_1 \cap E_2$ all are disjoint so we can write

$$m(E_1 \cup E_2) = m(E_1 \cap E_2') + m(E_2 \cap E_1') + m(E_1 \cap E_2) \longrightarrow *'$$

subtracting $*$ and $*'$ we get

$$m(E_1)+m(E_2)-m(E_1\cup E_2) = m(E_1\cap E_2)$$
$$\Rightarrow m(E_1\cup E_2) = m(E_1)+m(E_2)-m(E_1\cap E_2)$$

COROLLARY:-

If E_1, E_2, E_3 are measurable sets then

$$m(E_1\cup E_2\cup E_3) = m(E_1)+m(E_2)+m(E_3) - m(E_1\cap E_2) - m(E_2\cap E_3) - m(E_3\cap E_1) + m(E_1\cap E_2\cap E_3)$$

PROOF

As we know that if two sets E_1 and E_2 are measurable, then

$$m(E_1\cup E_2) = m(E_1)+m(E_2)-m(E_1\cap E_2) \longrightarrow (i)$$

Now put $E = E_1\cup E_2$ Then

$$E_1\cup E_2\cup E_3 = E\cup E_3$$

$$\Rightarrow m(E_1\cup E_2\cup E_3) = m(E\cup E_3)$$
$$= m(E)+m(E_3)-m(E\cap E_3)$$
$$= m(E_1\cup E_2)+m(E_3) - m((E_1\cup E_2)\cap E_3)$$
$$= m(E_1)+m(E_2) \ m(E_1\cap E_2)+m(E_3) - m[(E_1\cap E_3)\cup(E_2\cap E_3)]$$

using (i)

$$\Rightarrow m(E_1\cup E_2\cup E_3) = m(E_1)+m(E_2)+m(E_3)-m(E_1\cap E_2) - m(E_1\cap E_3) + m(E_2\cap E_3) - m(E_1\cap E_3\cap E_2\cap E_3)$$

Again by using (i)

$$\Rightarrow m(E_1\cup E_2\cup E_3) = m(E_1)+m(E_2)+m(E_3)-m(E_1\cap E_2) - m(E_2\cap E_3) - m(E_1\cap E_3) - m(E_1\cap E_2\cap E_3)$$

2014

PROPOSITION:-

Show that the cantor set C has Lebesgue measure zero.

PROOF:-

First we introduce the concept of cantor set. Let $I = [0,1]$ and $I_{11} =]\frac{1}{3}, \frac{2}{3}[$ be the middle third of I. We remove I_{11} from I. Further let $I_{21} =]\frac{1}{9}, \frac{2}{9}[$, $I_{22} =]\frac{7}{9}, \frac{8}{9}[$ be the middle third of the remaining intervals. After removing these two we have four closed intervals $[0, \frac{1}{9}]$, $[\frac{2}{9}, \frac{1}{3}]$, $[\frac{2}{3}, \frac{7}{9}]$, $[\frac{8}{9}, 1]$.
Let $I_{31}, I_{32}, I_{33}, I_{34}$ be the middle third of these four closed intervals. After removing these four closed → open intervals we have eight closed intervals.
Let $I_{41}, I_{42}, I_{43}, \ldots, I_{48}$ be the middle third of these remaining eight closed intervals. After removing these middle thirds we have remaining sixteen closed intervals.
Note that at 1st step 2^{1-1} open intervals i.e. I_{11} are removed.
At second step 2^{2-1} open intervals i.e. I_{21}, I_{22} are removed.
At third step 2^{3-1} open intervals i.e. $I_{31}, I_{32}, I_{33}, I_{34}$ are removed.

At 4^{th} step 2^{4-1} open intervals: I_{41}, $I_{42}, I_{43}, \ldots, I_{48}$ are removed.
Continuing this way at n^{th} step 2^{n-1} pairwise disjoint open intervals I_{nm}, $m = 1, 2, 3, \ldots, 2^{n-1}$ are removed.

Put $G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}$

then G is an open set being a union of open intervals.
Then as an open set is measurable so G is measurable
Complement of this set G w.r.t I is denoted by C and is called cantor set.
i.e $C = [0,1] \setminus G$ As complement of a measurable set is measurable so C is measurable
Further as G is open so $G' = C$ is closed

$$\text{Now } m(G) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right)$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} m(I_{nm})$$

$$= m(I_{11}) + [m(I_{21}) + m(I_{22})] + [m(I_{31}) + m(I_{32})] + \cdots$$

$$= \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots$$

$$= \frac{1/3}{1 - 2/3} = 1$$

Now $m([0,1]) = m(C) + m(G)$

$\because [0,1] = C \cup G$ and $C \cap G = \phi$

$\Rightarrow 1 - 0 = m(C) + 1$

$\Rightarrow m(C) = 0$

PROPOSITION:-

Let F be a subset of $\{0,1\}$ constructed in the same manner as the cantor set (also called cantor ternery set) except that each of the interval length removed at the nth step has length $\alpha . 3^{-n}$ with $0 < \alpha < 1$, then F is closed set and $m(F) = 1 - \alpha$

PROOF:

As in the construction of cantor set at the nth step we removed 2^{n-1} pairwise disjoint open intervals I_{nm}, $m = 0, 1, 2, 3, \ldots 2^{n-1}$ and each of length 3^{-n} but by given condition, here in the construction of F at the nth step we remove 2^{n-1} pairwise disjoint open intervals I_{nm}, $m = 1, 2, \ldots 2^{n-1}$ and each of length $0 < \alpha < 1$ so then

if $G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}$ and $F = [0,1] \setminus G$

then $m(G) = m(I_{11}) + [m(I_{21}) + m(I_{22})] + [m(I_{31}) + m(I_{32}) + m(I_{33}) + m(I_{34})] + \cdots$

$= \alpha/3 + 2\alpha/9 + 4\alpha/27 + \cdots$

$$= \alpha\left[\tfrac{1}{3} + \tfrac{2}{9} + \tfrac{4}{27} + \cdots\right]$$
$$= \alpha\left[\frac{1/3}{1-2/3}\right] = \alpha(1) = \alpha$$

so then $m([0,1]) = m(F) + m(G)$
$$\Rightarrow 1 = m(F) + \alpha$$
$$\Rightarrow m(F) = 1 - \alpha$$

This F is called generallized cantor set.

REMARK:

Every $x \in [0,1]$ can be written as $x = \sum_{n=1}^{\infty} a_n/2^n$, $a_n \in \{0,1\}$ and every $y \in C$, cantor set can be written as

$$y = \sum_{n=1}^{\infty} 2a_n/3^n \quad , \; a_n \in \{0,1\}$$

THEOREM:-

Prove that cantor set C is uncountable.

PROOF:-

let us define $F: [0,1] \to C$ by
$$f(x) = y$$

where $x = \sum_{n=1}^{\infty} a_n/2^n$

$$y = \sum_{n=1}^{\infty} 2a_n/3^n$$

$a_n \in \{0,1\}$, then

(i) f is 1—1 :-

Let $f(x_1) = f(x_2)$

where $x_1 = \sum_{n=1}^{\infty} a_n/2^n$ & $x_2 = \sum_{n=1}^{\infty} b_n/2^n$

$a_n, b_n \in \{0,1\}$

Now $f(x_1) = f(x_2)$

$\Rightarrow f\left(\sum_{n=1}^{\infty} a_n/2^n\right) = f\left(\sum_{n=1}^{\infty} b_n/2^n\right)$

$\Rightarrow \sum_{n=1}^{\infty} 2a_n/3^n = \sum_{n=1}^{\infty} 2b_n/3^n$

$\Rightarrow a_n = b_n$

$\Rightarrow \sum_{n=1}^{\infty} a_n/2^n = \sum_{n=1}^{\infty} b_n/2^n$

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is 1—1

(ii) f is onto:-

As for every $y \in \sum_{n=1}^{\infty} 2a_n/3^n = C$

we have $x = \sum_{n=1}^{\infty} a_n/2^n \in [0,1]$ s.t

$f(x) = y$

$\Rightarrow f$ is onto

$\Rightarrow f$ is bijective

$\Rightarrow C$ is uncountable.

Unterarad

SORRY YAR

INTERESTING FACTS:-

i- Measure of an open set is always non zero but measure of closed set might be zero e.g. Cantor set C is closed and $m(C) = 0$.

ii- If the measure of set is non zero then it is uncountable. But if a set is uncountable then its measure may be zero. for example Cantor set C is uncountable and has measure zero.

iii- Note that it might be possible that $A \sim B$ but both has different measure e.g. $\mathbb{R} \sim [0,1] \sim C$ and $m(\mathbb{R}) = \infty$ and $m[0,1] = 1$ and $m(C) = 0$

iv- Any finite set in $\mathbb{R}$ is closed and so is measurable.

BOREL SET:-

Definition:- The collection $\mathcal{B}$ of borel sets is defined to be a algebra generated by the collection of the intervals of the form $]a,b]$, $a,b \in \mathbb{R}$. The existence of $\mathcal{B}$ is guaranteed by "let G be a family of subsets of X, then there is a smallest σ-algebra containing G"
Since open interval $]a,b[= \bigcup_{n=1}^{\infty}]a, b-1/n]$

$\Rightarrow \beta$ contains all open intervals.

Similarly, we may replace $]a,b]$ in the definition of β (Borel set) by $[a,b[$, $]a,\infty[$, $]-\infty,b[$, $]-\infty,b]$ etc.

2015

THEOREM:- The borel σ-algebra β is generated by each of the following collection of sets.

1). The collection C_1 of all closed subsets of $\mathbb{R}$.

2). The collection C_2 of all subintervals of the form $]-\infty,b]$

3). The collection C_3 of all subintervals of the form $]a,b]$.

PROOF:

Let the σ-algebra generated by C_1, C_2, C_3 be denoted by $\beta_1, \beta_2, \beta_3$ respectively. Further by definition of Borel σ-algebra β, β contain the family of open sets and is closed under complementation. Now, To prove

$$\beta = \beta_1 = \beta_2 = \beta_3$$

As β contains the family of open sets and is closed under complementation, so β also contains closed sets.

($\because$ Complement of an open set is closed)

$\Rightarrow \mathcal{B}_1 \subseteq \mathcal{B}$

Now $]-\infty, b] =]b, \infty['$

$\Rightarrow]-\infty, b]$ are closed sets for $b \in \mathbb{R}$ and so belongs to $\mathcal{B}_1$

$\Rightarrow \mathcal{B}_2 \subseteq \mathcal{B}_1$

Now,

$]a, b] =]-\infty, b] \cap]a, \infty[$

$=]-\infty, b] \cap]-\infty, a]'$

$\Rightarrow]a, b] \in \mathcal{B}_2$

$\Rightarrow \mathcal{B}_3 \subseteq \mathcal{B}_2$

Now $]a, b[= \bigcup_{n=1}^{\infty}]a, b - \frac{1}{n}] \in \mathcal{B}_3$

$\Rightarrow \mathcal{B} \subseteq \mathcal{B}_3$

$\Rightarrow \mathcal{B} \subseteq \mathcal{B}_3 \subseteq \mathcal{B}_2 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}$

$\Rightarrow \mathcal{B} = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3$

THEOREM:-

Every Borel set is measurable.

PROOF:-

We know that every open set is measurable, so it means $\mathcal{M}$ contains all open sets. But $\mathcal{B}$ is the smallest σ-algebra containing all open sets.

Hence $\mathcal{B} \subseteq \mathcal{M}$

$\Rightarrow$ Every Borel set is measurable.

DEFINITION:-

The restriction of Lebesgue measure m to $\mathcal{B}$ is called Borel measure all the triplet $(\mathbb{R}, \mathcal{B}, m)$ is known as Borel measure space.

THEOREM:-

Any Singleton set, finite set and countable set is a Borel set, with Borel measure zero.

PROOF:-

To prove the theorem, It is sufficient to prove that any singlton set is a Borel set with Borel measure zero because the remaining part of the theorem is obvious by the fact that $\mathcal{B}$ is σ-algebra and m is finite and sigma additive.

let a be the real number then we have $\{a\} = \bigcap_{n=1}^{\infty}]a - 1/n, a]$

Then $\{a\} \in \mathcal{B}$.

$\Rightarrow \{a\}$ is Borel set.

let $E_n =]a - 1/n, a]$

Then E_n is decreasing sequence and

$m(E_1) = m(]a-1, a[) = a-(a-1) = 1 < \infty$

So $m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n) = \lim_{n\to\infty} m(]a-1/n, a])$

$= \lim_{n\to\infty} [a-(a-1/n)]$

$= \lim_{n\to\infty} (1/n) = 0$

$\Rightarrow m(\{a\}) = 0$

EXISTANCE OF NON MEASURABLE SETS:-

We have seen that

1- Every finite set, every countable set, every interval is measurable.

2- ϕ, $\mathbb{R}$ are measurable.

3- Every open set, Every closed set is measurable.

4- If E is measurable, then E' is measurable.

5- F_σ and G_δ set is measurable.

6- Cantor set "C" is measurable.

7- Union and intersection of a sequence of measurable sets is measurable.

It seems from the above that every subset of $\mathbb{R}$, we think is measurable. But the fact is that it is not true and non measurable sets exists.

Definition: Let $A = [0,1[$ and $x, y \in A$, we denote and define sum modulu of x and y by

$$x \dotplus y = \begin{cases} x+y & \text{if } x+y < 1 \\ x+y-1 & \text{if } x+y \geq 1. \end{cases}$$

Definition:- let $A = [0,1[$, and $E \subseteq A$ and $y \in A$, then sum modulo of E and y is denoted and defined by

$$E \dotplus y = \{x \dotplus y : x \in E\}$$

2012
2013

LEMMA:- Let $A = [0,1[$, $E \subseteq A$, $y \in A$ and E is measurable set then $E \dotplus y$ is measurable and $m(E \dotplus y) = m(E)$

PROOF:-

Let $E_1 = E \cap [0, 1-y[$ and $E_2 = E \cap [1-y, 1[$.
Then E_1 and E_2 are measurable because E, $[0, 1-y[$, $[1-y, 1[$ are measurable and intersection of measurable sets is measurable.

Also it is clear that $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \phi$, so then by the finite additivity of m, $m(E) = m(E_1) + m(E_2)$ $\longrightarrow *$

Now, if $y_1 \in E_1$, then $y_1 \in E$, $y_1 \in [0, 1-y[$

$\Rightarrow y_1 \in [0, 1-y[\Rightarrow y_1 < 1-y \Rightarrow y_1 + y < 1$

$\Rightarrow E_1 \dotplus y = E_1 + y$

If $y_2 \in E_2 \Rightarrow y_2 \in E$ and $y_2 \in [1-y, 1[$

$\Rightarrow y_2 \in E_2 \Rightarrow y_2 \in E$ and $y_2 \in [1-y, 1[$

$\Rightarrow y_2 \in [1-y, 1[\Rightarrow y_2 \geq 1-y \Rightarrow y_2 + y \geq 1$

$\Rightarrow E_2 \dotplus y = E_2 + (y-1)$

Now as E_1 and E_2 are measurable and translation of measurable set is measurable so E_1+y, $E_2+(y-1)$ are measurable sets.

$\Rightarrow E_1 \dot{+} y$ and $E_2 \dot{+} y$ are measurable

Now, as m is translation invariant, so

$m(E_1 \dot{+} y) = m(E_1+y) = m(E_1)$

and $m(E_2 \dot{+} y) = m(E_2+(y-1)) = m(E_2)$

Also as $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \phi$, so

$(E_1 \dot{+} y) \cup (E_2 \dot{+} y) = E \dot{+} y$

and $(E_1 \dot{+} y) \cap (E_2 \dot{+} y) = \phi$

$\Rightarrow m(E \dot{+} y) = m(E_1) + m(E_2) = m(E)$

$\Rightarrow m(E \dot{+} y) = m(E)$

THEOREM:-

2016 **Demonstrate the existance of non Lebesgue measurable sets.**

PROOF:-

Let $A = [0,1[$ and for $x, y \in A$, define a relation $\sim$ on A by $x \sim y$ iff $x-y$ is a rational number. Then

i. As for all $x \in A$, $x-x$ is rational $\Rightarrow x \sim x$

$\Rightarrow$ "$\sim$" is reflexive

ii. If $x \sim y$, $x, y \in A$, then $x-y$ is rational

$\Rightarrow y-x$ is rational $\Rightarrow y \sim x \Rightarrow$ "$\sim$" is symmetric

iii. let $x, y, z \in A$ s.t. $x \sim y$ and $y \sim z$

$\Rightarrow x-y$ and $y-z$ are rational

$\Rightarrow x-y+y-z$ is rational.

$\Rightarrow z - x$ is rational $\Rightarrow \sim$ is transitive.
$\Rightarrow$ "$\sim$" is equivalance relation.
Hence $\sim$ partitions A into disjoint equivalance classes s.t. each equivalance class contains those elements of A which differ by a rational number.
let E be the collection of all those equivalance classes Then $E \neq \phi$, because at least equivalance class determined by $0 \in A$ is in E.
Then by the axiom of choice, there exists a choice function on E and so we can find a set P which contain exactly one point from each equivalance class. Now,
1) we express A as countable union of disjoint subset of itself. For this consider an enumeration $\{r_i : i = 0, 1, 2, \ldots\}$ of rational numbers in A with $r_0 = 0$ and put $P_i = P \dotplus r_i$, then clearly $P_0 = P$
Then, if $x \in A$, then x is in some equivalance class in the collection E.
Then x is equivalant to some element p_i of P.

$\Rightarrow x - p_i$ is a rational number
$\Rightarrow x - p_i = r_i$, for some rational r_i
$\Rightarrow x = p_i + r_i \in P \dotplus r_i = P_i$
$\Rightarrow x \in P_i$, for some i
$\Rightarrow x \in \bigcup_i P_i \Rightarrow A \subseteq \bigcup_i P_i$ but

$\bigcup_i P_i \subseteq A \quad \Rightarrow \quad A = \bigcup_i P_i$

2) Now we prove that all P_i's are pair-wise disjoint.
Suppose on the contrary that there are some $i \neq j$ and $P_i \cap P_j \neq \phi$

$\Rightarrow x \in P_i \cap P_j \Rightarrow x \in P_i$ and $x \in P_j$

$\Rightarrow x \in P \dot{+} r_i$ and $x \in P \dot{+} r_j$

$\Rightarrow x = p_i + r_i$ & $x = p_j + r_j$

where $p_i, p_j \in P$

$\Rightarrow p_i + r_i = p_j + r_j$

$\Rightarrow p_i - p_j = r_j - r_i$

$\Rightarrow p_i - p_j$ is rational

$\Rightarrow p_i - p_j$ belongs to the same equivalance class.

which is a contradiction because P contains exactly one element from each equivalance class

So our supposition is wrong and hence all P_i's are pairwise disjoint *

Now we prove that the set P is non Lebesgue measurable

Suppose on the contrary that P is measurable. then $P \dot{+} r_i$ is measurable

$\Rightarrow P_i$ is measurable, for each i.

Now,

$m(A) = m(\bigcup_i P_i) \qquad \because A = \bigcup_i P_i$

$= \sum_{i=1}^{\infty} m(P_i) \qquad \because m$ is a additive

$= \sum_{i=1}^{\infty} m(P \dot{+} r_i) \qquad \because P_i = P \dot{+} r_i$

$$= \sum_{i=1}^{\infty} m(P) \qquad \because m(P+r_i) = m(P)$$

$$\Rightarrow m(A) = \sum_{i=1}^{\infty} m(P)$$

Now by assumption P is measurable

$\Rightarrow m(P) \geq 0$

If $m(P) > 0$, then $m(A) = \sum_{i=1}^{\infty} m(P) = \infty$

$\Rightarrow 1 = \infty \qquad \because m(A) = 1$

which is not possible, so

$m(P) \not> 0$

If $m(P) = 0$, then $m(A) = \sum_{i=1}^{\infty} m(P) = 0$

$\Rightarrow 1 = 0 \qquad \because m(A) = 1$

which is not possible $\Rightarrow m(P) \neq 0$

$\Rightarrow P$ is measurable and $m(P) \not\geq 0$, which is a contradiction, so our supposition is wrong.

Hence P is not measurable.

THEOREM:-

If E is measurable set and P is the non-measurable set s.t. $E \subset P \subset [0,1[$, then $m(E) = 0$

PROOF:-

Consider the enumeration $\{r_i : i = 0, 1, 2, \ldots\}$ of rational numbers with $r_0 = 0$. Then if $P_i = P + r_i$, then all P_i's are pairwise disjoint and $A = \bigcup_i P_i$

Put $E_i = E + r_i$

$E_i = E \dot{+} r_i \subseteq P \dot{+} r_i = P_i$ and also E_i's are pairwise disjoint and $\bigcup_i E_i \subseteq \bigcup_i P_i = A$ and also $m(E_i) = m(E)$

Thus $m(A) \geq m(\bigcup_i E_i)$

$$= \sum_i m(E_i) = \sum_i m(E)$$

As E is measurable so $m(E) \geq 0$

If $m(E) > 0$, then $\sum_i m(E) = \infty$

$$\Rightarrow m(A) = \infty$$

which is a contradiction because $m(A) = m([0,1[) = 1$

Hence $m(E) \not> 0$ and thus $m(E) = 0$.

THEOREM:-

If μ is a translation invariant measure defined on the σ-algebra containing the set P, then $\mu([0,1[) = \infty$ or $\mu([0,1[) = 0$

PROOF:-

Let $A = [0,1[$ and define relation "$\sim$" on A by $x, y \in A \Rightarrow x \sim y$ if $x - y$ is rational number .... upto $*$

Now $\mu(A) = \mu([0,1[) = \mu(\bigcup_i P_i)$

$$= \sum_i \mu(P_i) = \sum_i \mu(P \dot{+} r_i) = \sum_i \mu(P)$$

$$\Rightarrow \mu([0,1[) = \sum_i \mu(P)$$

If $\mu(P) = 0$, then $\mu([0,1[) = 0$

If $\mu(P) > 0$, then $\mu([0,1[) = \infty$

Hence either $\mu([0,1[)$ is either zero or ∞.

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SORRY YAR

THEOREM:-

2020, 2019

Let A be a Lebesgue measurable set with $m(A) > 0$, then there exist $E \subseteq A$ s.t. E is not Lebesgue measure.

Proof:- Put $A = \bigcup_{n=1}^{\infty} (A \cap [-n, n])$.

As $m(A) > 0$, so there is a +ve integer n_0 such that $m(A \cap [-n_0, n_0]) > 0$.

Take $B = A \cap [-n_0, n_0]$ & for any $x \in B$; we define

$$B_x = \{ y \in B : y - x \text{ is rational} \}$$

Clearly $B = \bigcup_{x \in B} B_x$

For $x_1, x_2 \in B$, $B_{x_1} = B_{x_2}$ only if $x_1 - x_2$ is rational as follows:

If $a \in B_{x_1} \Rightarrow a - x_1$ is rational, then $a - x_2 = (a - x_1) + (x_1 - x_2)$ is rational

$\Rightarrow a \in B_{x_2} \Rightarrow B_{x_1} \subseteq B_{x_2}$. Similarly $B_{x_2} \subseteq B_{x_1} \Rightarrow B_{x_1} = B_{x_2}$.

Also for $x_1 \neq x_2$, $B_{x_1} \cap B_{x_2} = \phi$.

Because otherwise $a \in B_{x_1} \cap B_{x_2}$

$\Rightarrow a \in B_{x_1}$ & $a \in B_{x_2}$

$\Rightarrow a - x_1$ & $a - x_2$ are rational.

$\Rightarrow -(a - x_1) + (a - x_2) = x_1 - x_2$ is rational.

$\Rightarrow B_{x_1} = B_{x_2} \Rightarrow x_1 = x_2$ A cont.

Now by the axiom of choice there is a set $E \subseteq B = \bigcup_{x \in B} B_x$ containing exactly exactly one point from each of the distinct sets $\{B_x\}$. Now we claim that the set E is not Lebesgue measurable.

For this consider the enumeration $\{r_i : i = 1, 2, 3, ---\}$ of rational numbers

in $[-2n_0, 2n_0]$, & put $E_i = E + r_i$.
If $i \neq j$, then $r_i \neq r_j$, for $x \in E_i \cap E_j$
we have $x = x_1 + r_i$, $x = x_2 + r_j$
where $x_1, x_2 \in E$. Since $r_i \neq r_j$
& $x_1 + r_i = x_2 + r_j$, so $x_1 \neq x_2$
$\Rightarrow x_1$ & x_2 are in different classes
$\Rightarrow x_1 - x_2$ is irrational.
But $x_1 - x_2 = r_j - r_i \Rightarrow x_1 - x_2$ is rational.
A contradiction. So for $i \neq j$, $E_i \cap E_j = \phi$
Also

$$\bigcup_{i=1}^{\infty}(E + r_i) = \bigcup_{i=1}^{\infty} E_i \subset [-3n_0, 3n_0]$$

because $E \subseteq [-n_0, n_0)$ & $r_i \in [-2n_0, 2n_0]$.
If $x \in B$, then x is in some class B_e. By definition of E, e must be in E. Thus $x - e = r_j$ is a rational in $[-2n_0, 2n_0]$ i.e.
$x = e + r_j \in E + r_j = E_j \Rightarrow B \subseteq E_j$
Hence $B \subseteq \bigcup_{i=1}^{\infty} E_i$. Now if E is measurable, then each E_i is measurable & $m(E_i) = m(E)$, as m is translation invariant. Thus

$$0 < m(B) \leq m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i) \cancel{\leq 6n_0}$$

$$= \sum_{i=1}^{\infty} m(E) \leq 6n_0$$

Now if $m(E) = 0$ then $m(B) = 0$ which is not possible, as $m(B) > 0$.
If $m(E) > 0$, then $\infty = \sum m(E_i)$

$$= \sum m(E) \leq 6n_0$$

which is again contradiction.
Hence E is not measurable.

REMARKS:-

(i) Every set of Positive measure in $\mathbb{R}$ contains at least one non measurable set.

(ii) Every subset of a measurable set need not to be measurable and also every subset of a non measurable set need not to be non-measurable.

(iii) Every measurable subset of a non measurable set has measure zero.

(iv) Complement of a non measurable set is non measurable. However, union and intersection of two non measurable sets need not to be non measurable

For example:

$P \cup P' =]0,1[$ and $P \cap P' = \phi$

shows that union and intersection of non measurable sets might be measurable.

(v) Lebesgue outer measure of a non measurable set is necessarily +ve.

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ONCE AGAIN

Measurable Functions:-

Let $(X, \mathcal{A}, \mu)$ be a measure space and $E \in \mathcal{A}$ then an extended real valued function f on E is said to be measurable function if for each $\alpha \in \mathbb{R}$, $\{x \in E : f(x) > \alpha\} \in \mathcal{A}$. In other words, if $D \subseteq \mathbb{R}$ and D is measurable then a function $f : D \to \overline{\mathbb{R}}$ is measurable if for all $\alpha \in \mathbb{R}$, $\{x \in D : f(x) > \alpha\}$ is measurable.

Theorem:-

Let f be an extended real valued function defined on a measurable set D, then the following are equivalent.

(i) f is measurable.

(ii) $\{x \in D : f(x) \geq \alpha\}$ is measurable $\forall \alpha \in \mathbb{R}$

(iii) $\{x \in D : f(x) < \alpha\}$ " " " "

(iv) $\{x \in D : f(x) \leq \alpha\}$ " " " "

(v) Moreover (i) $\to$ (iv) implies that $\{x \in D : f(x) = \alpha, \forall \alpha \in \overline{\mathbb{R}}\}$ is measurable.

Proof:-

(i) $\Rightarrow$ (ii)

i.e. f is measurable and To prove for all $\alpha \in \mathbb{R}$, $\{x \in D : f(x) \geq \alpha\}$ is measurable.

As f is measurable, so $\{x \in D : f(x) > \alpha\}$; for all $\alpha \in \mathbb{R}$ is measurable.

We claim that

$$\{x \in D : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \in D : f(x) > \alpha - 1/n\}$$

Let $y \in \{x : f(x) \geq \alpha\}$

$\Rightarrow f(y) \geq \alpha$

$\Rightarrow f(y) > \alpha - 1/n$, for all $n \in \mathbb{N}$

$\Rightarrow y \in \{x : f(x) > \alpha - 1/n\}$, for all n

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$

$\Rightarrow \{x : f(x) \geq \alpha\} \subseteq \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$

working backward, we have.

$\bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\} \subseteq \{x : f(x) \geq \alpha\}$

$\Rightarrow \{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$

As given $\{x : f(x) > \alpha - 1/n\}$ is measurable and countable intersection of measurable sets is measurable so $\bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - 1/n\}$ is measurable.

$\Rightarrow \{x : f(x) \geq \alpha\}$ is measurable.

(ii) $\Rightarrow$ (iii)

i.e. given $\{x : f(x) \geq \alpha\}$ is measurable and to prove $\{x : f(x) < \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$.

Note that $\{x : f(x) < \alpha\} = D \setminus \{x : f(x) \geq \alpha\}$
$= D \cap \{x : f(x) \geq \alpha\}'$

As R.H.S is measurable, so L.H.S is measurable.

(iii) $\Rightarrow$ (iv)

i.e given $\{x : f(x) < \alpha\}$ is measurable. To prove $\{x : f(x) \leq \alpha\}$ is measurable.

$$\{x : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + 1/n\}$$

As $\{x : f(x) < \alpha + 1/n\}$ is measurable and countable intersection of measurable set is measurable so R.H.S is measurable

$\Rightarrow$ L.H.S is measurable.

(iv) $\Rightarrow$ (i)

i.e. given $\{x \in D : f(x) \leq \alpha\}$ is measurable

To prove f is measurable.

As $\{x : f(x) > \alpha\} = D \setminus \{x : f(x) \leq \alpha\}$

Since R.H.S is measurable

$\Rightarrow f$ is measurable.

(v) let $\alpha \in \mathbb{R}$, then

$$\{x : f(x) = \alpha\} = \{x : f(x) \geq \alpha\} \cap \{x : f(x) \leq \alpha\}.$$

Since R.H.S is measurable so L.H.S is measurable.

If $\alpha = \infty$, then $\{x : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) > n\}$

If $\alpha = -\infty$, then $\{x : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x : f(x) < -n\}$

$\Rightarrow \{x : f(x) = \alpha\}$ is measurable for $\alpha = -\infty, \infty$

$\Rightarrow \{x : f(x) = \alpha\}$ is measurable for every extended real number α i.e. $\alpha \in \overline{\mathbb{R}}$

REMARKS:-

(i) It is obvious that if a result holds for extended real valued function, then it holds in particular for real valued function.

(ii) From previous theorem (Part : v) It is clear that extended real valued constant function is measurable.

However, It will be interested to note that for $f(x) = c$, where $c \in \overline{\mathbb{R}}$, then the measurability of f follows from

$$\{x \in D : f(x) > \alpha\} = \begin{cases} D & \text{if } \alpha < c \\ \phi & \text{if } \alpha \geq c \end{cases}$$

and the measurability of D and ϕ.

(iii) In previous theorem (Part: v) we have shown that if f is measurable function on D, then the set $\{x \in D : f(x) = \alpha\}$ is measurable for each real number α. However conversely if $\{x \in D : f(x) = \alpha\}$ is measurable set for each $\alpha \in \mathbb{R}$ then f is not necessarily measurable for example:

Exp: let P be a non measurable +ve subset of $\mathbb{R}$, then also P' is non measurable.

Now, Put $A = \{x \in D : x > 0\}$ and
$B = \{x \in D : x \leq 0\}$ then $A \cup B = D$
and $A \cap B = \phi$.
Now, let

$$g : A \longrightarrow P \quad \text{and} \quad h : B \longrightarrow P'$$

be any two bijective function and define

$$f : D \longrightarrow \mathbb{R} \text{ by } f(x) = \begin{cases} g(x) & \text{if } x \in A \\ h(x) & \text{if } x \in B \end{cases}$$

Then f is extension of both g and h also obviously bijective function
Then for any real number α, The set $\{x \in D : f(x) = \alpha\}$ contains exactly one point and so it is measurable. ?
But note that the set $\{x \in D : f(x) > 0\}$ being the same as P is non measurable. Hence f is not a measurable function.

Definition:-

Let f be an extended real valued function defined on any set A. Then the +ve part f^+ and the -ve part f^- of f are the extended real valued functions defined by

$$f^+(x) = \text{Max}\{f(x), 0\} = f \vee 0$$

and $f^-(x) = \text{Max}\{-f(x), 0\} = -f \vee 0$ for all $x \in A$.

THEOREM:-

If f is an extended real valued function, then,

(i) $f = f^+ - f^-$ (ii) $|f| = f^+ + f^-$

PROOF:-

Here arises the following cases:

Case-I:-

If $f(x) = 0$, for all x

Then $f^+(x) = Max\{f(x), 0\} = 0$

$$f^-(x) = Max\{-f(x), 0\} = 0$$

$$\Rightarrow f^+(x) - f^-(x) = 0 - 0 = 0 = f(x)$$

$$\Rightarrow f(x) = f^+(x) - f^-(x)$$

$$\Rightarrow f = f^+ - f^-$$

Case-II:-

If $f(x) > 0$, then

$f^+(x) = Max\{f(x), 0\} = f(x)$

$f^-(x) = Max\{-f(x), 0\} = 0$

$$\Rightarrow f^+(x) - f^-(x) = f(x) - 0 = f(x)$$

$$\Rightarrow f^+ - f^- = f$$

Case-III:-

If $f(x) < 0$, then

$f^+(x) = Max\{f(x), 0\} = 0$

$f^-(x) = \text{Max}\{-f(x), 0\} = -f(x)$
$\because -f(x) > 0$

Thus

$f^+(x) - f^-(x) = 0 - (-f(x)) = f(x)$

$\Rightarrow f^+ - f^- = f$

Hence combining all the cases

$f = f^+ - f^-$

(ii) $|f| = f^+ + f^-$

Sol

Case - I :-

if $f(x) = 0 \Rightarrow |f(x)| = 0, \forall x \in A$

Then $f^+(x) = \text{Max}\{f(x), 0\} = 0$

$f^-(x) = \text{Max}\{-f(x), 0\} = 0$

$\Rightarrow f^+(x) + f^-(x) = 0 + 0 = 0 = |f(x)|$

$\Rightarrow |f| = f^+ + f^-$

Case - II :-

if $f(x) > 0$, then $|f(x)| = f(x)$

Now,

$f^+(x) = \text{Max}\{f(x), 0\} = f(x)$

$f^-(x) = \text{Max}\{-f(x), 0\} = 0$

so $f^+(x) + f^-(x) = f(x) + 0 = f(x) = |f(x)|$

$\Rightarrow |f| = f^+ + f^-$

Case- III :-

when $f(x) < 0$ then $|f(x)| = -f(x)$

Now $f^+(x) = \max\{f(x), 0\} = 0$

$f^-(x) = \max\{-f(x), 0\} = -f(x)$

Now, $f^+(x) + f^-(x) = 0 + (-f(x))$.

$= -f(x)$

$= |f(x)|$

$\Rightarrow |f| = f^+ + f^-$

Hence combining all the cases:

$\Rightarrow |f| = f^+ + f^-$

THEOREM:-

Let f and g be two real valued measurable functions on some measurable domain D and $c \in \mathbb{R}$, then

(i) $f + c$ (ii) cf (iii) $f + g$ $\frac{f}{g}$

(iv) $f - g$ (v) fg (vi) $f \setminus g$, $g \neq 0$

(vii) $f \vee g$ (viii) $f \wedge g$ (ix) $|f|$ are

all measurable functions.

PROOF:-

(i) $f+c$ is measurable.

Let $\alpha \in \mathbb{R}$, then as f is measurable then

$\{x \in D : f(x) > \alpha\}$ is measurable for all $\alpha \in \mathbb{R}$.

Now, $\{x \in D : (f+c)(x) > \alpha\}$
$= \{x \in D : f(x) > \alpha - c\}$

$\Rightarrow \{x \in D : (f+c)(x) > \alpha\} = \{x \in D : f(x) > \alpha - c\}$

As R.H.S is measurable, so L.H.S is also measurable.

$\Rightarrow f+c$ is measurable.

(ii) cf is measurable.

Here arises the following cases:

Case-I:

If $c=0$, then $(cf)(x) = c\,f(x)$
$= 0 \cdot f(x) = 0$

$\Rightarrow cf$ is then a constant function and hence is measurable.

Case-II:

If $c>0$, then for any $\alpha \in \mathbb{R}$

$\{x \in D : (cf)(x) > \alpha\} = \{x \in D : c\,f(x) > \alpha\}$

$= \{x \in D : f(x) > \alpha/c\}$

$\Rightarrow \{x \in D : (cf)(x) > \alpha\} = \{x \in D : f(x) > \alpha/c\}$

R.H.S is measurable ($\because f$ is measurable)
so L.H.S is measurable.

$\Rightarrow cf$ is measurable.

Case - III :-
If $c < 0$, then for any $\alpha \in \mathbb{R}$

$\{x \in D : (cf)(x) > \alpha\} = \{x \in D : f(x) < \alpha/c\}$

As R.H.S is measurable
($\because f$ is measurable)

So L.H.S is measurable.
$\Rightarrow cf$ is measurable.

Hence combining all the cases
cf is measurable.

(iii) $f + g$ is measurable

To prove $f + g$ is measurable while given f and g are measurable functions.

Let $\alpha \in \mathbb{R}$, be any real number
Now $(f+g)(x) > \alpha \Rightarrow f(x) + g(x) > \alpha$

$\Rightarrow f(x) > \alpha - g(x)$

Then by rational density theorem of real analysis, there exist some rational number γ such that

$$f(x) > \gamma > \alpha - g(x)$$

$\Rightarrow f(x) > \gamma$ and $\gamma > \alpha - g(x)$

$\Rightarrow f(x) > \gamma$ and $g(x) > \alpha - \gamma$

Now, we show that

$$\{x \in D : (f+g)(x) > \alpha\} = \bigcup_{\gamma \in \mathbb{Q}} \left[\{x \in D : f(x) > \gamma\} \cap \{x \in D : g(x) > \alpha - \gamma\} \right]$$

let $y \in \{x \in D : (f+g)(x) > \alpha\}$

$\Rightarrow (f+g)(y) > \alpha$

$\Rightarrow f(y) + g(y) > \alpha$

$\Rightarrow f(y) > \gamma$ and $g(y) > \alpha - \gamma$ for some $\gamma \in \mathbb{Q}$

$\Rightarrow y \in \{x \in D : f(x) > \gamma\} \cap \{x \in D : g(x) > \alpha - \gamma\}$

$\Rightarrow y \in \bigcup_{\gamma \in \mathbb{Q}} \left[\{x \in D : f(x) > \alpha\} \cap \{x \in D : g(x) > \alpha - \gamma\} \right]$

$\Rightarrow \{x \in D : (f+g)(x) > \alpha\} \subseteq \bigcup_{\gamma \in \mathbb{Q}} \left[\{x \in D : f(x) > \alpha\} \cap \{x \in D : g(x) > \alpha - \gamma\} \right] \rightarrow (1)$

Working backward we get the converse

result.

Hence

$$\{x \in D : (f+g)(x) > \alpha\} = \bigcup_{r \in Q} \left[\{x \in D : f(x) > r\} \cap \{x \in D : g(x) > \alpha - r\} \right]$$

As R.H.S is measurable
(∵ f and g are measurable and intersection of two measurable sets is measurable, also countable union of two measurable sets is measurable)

⇒ L.H.S is also measurable
⇒ $f+g$ is measurable.

(iv) $f-g$ is measurable:

As f and g are measurable.
⇒ $(-1)g$ is measurable.

⇒ $-g$ is measurable
and sum of two measurable functions is measurable
⇒ $f+(-g)$ is measurable

⇒ $f-g$ is measurable.

(v) f^2 is measurable:

To prove f^2 is measurable.
let $\alpha \in \mathbb{R}$, then

a–

if $\alpha < 0$, then

$\{x \in D : f^2(x) > \alpha\} = D$
and is measurable
$\Rightarrow f^2$ is measurable.

b. If $\alpha \geq 0$, then

$\{x \in D : f^2(x) \geq \alpha\} = \{x \in D : f(x) > \sqrt{\alpha}\} \cup \{x \in D : f(x) < -\sqrt{\alpha}\}$?

Since f is measurable so

$\{x \in D : f(x) > \sqrt{\alpha}\}$ and $\{x \in D : f(x) < -\sqrt{\alpha}\}$ is measurable and union of measurable sets is measurable
$\Rightarrow$ R.H.S is measurable.
So, L.H.S is also measurable.
$\Rightarrow f^2$ is measurable.

fg is measurable:

As $fg = \frac{1}{4}\{(f+g)^2 - (f-g)^2\} \longrightarrow$ (i)

As f and g are measurable functions
$\Rightarrow f+g$ and $f-g$ are measurable

$\Rightarrow (f+g)^2$ and $(f-g)^2$ are measurable.
($\because f$ is measurable $\Rightarrow f^2$ is measurable)

$\Rightarrow (f+g)^2 - (f-g)^2$ is measurable
($\because$ if f and g are measurable $\Rightarrow f-g$ is measurable.)

$\Rightarrow \frac{1}{4}\{(f+g)^2-(f-g)^2\}$ is measurable

$\because$ if f is measurable
$\Rightarrow cf$ is measurable

$\Rightarrow$ R.H.S of equ (i) is measurable

$\Rightarrow$ L.H.S is measurable
$\Rightarrow fg$ is measurable.

(vi) f/g is measurable.

As given $g(x) \neq 0$

So $\frac{1}{g(x)}$ is defined

let $\alpha \in \mathbb{R}$, then

$$\{x\in D: \frac{1}{g(x)}>\alpha\} = \begin{cases} \{x\in D: g(x)>0\}, & \text{if } \alpha=0 \\ \{x\in D: g(x)>0\}\cap\{x\in D: g(x)<1/\alpha\}, & \text{if } \alpha>0 \\ \{x\in D: g(x)>0\}\cup\{x\in D: g(x)<0\}\cap \\ \{x\in D: g(x)<1/\alpha\}, & \text{if } \alpha<0 \end{cases}$$

As R.H.S is measurable
($\because g$ is measurable)

So L.H.S is also measurable
$\Rightarrow 1/g$ is measurable
also f is measurable

$\Rightarrow f \cdot \frac{1}{g}$ is measurable

$\left(\because \text{if } f \text{ and } g \text{ are measurable} \Rightarrow fg \text{ is measurable} \right)$

$\Rightarrow f/g$ is measurable.

(vii) $f \vee g$ is measurable.

As $\{x : (f \vee g)(x) > \alpha\} = \{x : f(x) > \alpha\} \cup \{x : g(x) > \alpha\}$

Since R.H.S is measurable so L.H.S is also measurable.

$\Rightarrow f \vee g$ is measurable.

(viii) $f \wedge g$ is measurable.

As $\{x : (f \wedge g)(x) > \alpha\} = \{x : f(x) > \alpha\} \cap \{x : g(x) > \alpha\}$

Since R.H.S is measurable so L.H.S is measurable

$\Rightarrow f \wedge g$ is measurable.

IX $|f|$ is measurable.

As $|f| = f^+ + f^-$

As $f^+ = f \vee 0$

Since f is measurable. 0 i.e. zero function which maps all elements on zero. i.e. constant function and constant function is measurable

$\Rightarrow 0$ is measurable

$\Rightarrow f \vee 0$ is measurable.

$\Rightarrow f^+$ is measurable

also $(-1)f$ is measurable

$\Rightarrow -f$ is measurable

$\Rightarrow -f \vee 0$ is measurable

$\Rightarrow f^-$ is measurable

$\Rightarrow f^+ + f^-$ is measurable

$\Rightarrow |f|$ is measurable.

REMARK:-

Converse of the theorem "If f is measurable, then so is $|f|$" is not true in general.

e.g.

Exp:

Let E be a non measurable set and χ_E is defined as

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

Now let $f(x) = \chi_E(x) - \frac{1}{2}$

Then f is not a measurable function

because $\{x : f(x) > 0\} = E$ and E is non measurable.

$\Rightarrow f$ is non measurable

Now $|f(x)| = \frac{1}{2}$

Then $|f|$ is measurable
($\because$ constant function is measurable)

THEOREM:-
Let f be an extended real valued measurable function defined on measurable set D. Let A be a measurable subset of D then restriction of f to A is also measurable.

PROOF:

Let g be the restriction of f on $A \subseteq D$
Then for all $x \in A$, $g(x) = f(x)$
Now, let $\alpha \in \mathbb{R}$

$$\{x \in A : g(x) > \alpha\} = \{x \in A : f(x) > \alpha\}$$
$$\subseteq \{x \in D : f(x) > \alpha\}$$

$$\Rightarrow \{x \in A : g(x) > \alpha\} \subseteq \{x \in D : f(x) > \alpha\}$$
$$\Rightarrow \{x \in A : g(x) > \alpha\} = A \cap \{x \in D : f(x) > \alpha\}$$

As given A is measurable and also f is measurable

$\Rightarrow$ R.H.S is measurable
$\Rightarrow$ L.H.S is measurable
$\Rightarrow g$ is measurable.

COROLLARY:-

Let D and E be the two measurable sets and f is a function with

domain $D \cup E$, Then f is measurable iff restriction of f to D and E are measurable.

PROOF:-

Suppose restriction of f to D and E are measurable.
To prove f is measurable with domain $D \cup E$.

Now, $D \cup E$ is measurable being the union of two measurable sets.
let

$$\{x \in D \cup E : f(x) > \alpha\} = \{x \in D : f(x) > \alpha\} \cup \{x \in E : f(x) > \alpha\}$$

As R.H.S is measurable So, L.H.S is measurable.

$\Rightarrow$ f is measurable with domain $D \cup E$

Conversely Suppose f is measurable with domain $D \cup E$. To prove restriction of f to D and E are measurable.

As $D \subseteq D \cup E$
Then

$$\{x \in D : f(x) > \alpha\} \subseteq \{x \in D \cup E : f(x) > \alpha\}$$

$$\Rightarrow \{x \in D : f(x) > \alpha\} = D \cap \{x \in D \cup E : f(x) > \alpha\}$$

As D and $\{x \in D \cup E : f(x) > \alpha\}$ are measurable $\Rightarrow$ R.H.S is measurable

$\Rightarrow$ L.H.S is measurable

$\Rightarrow$ restriction of f to D is measurable.

Similarly, we can prove that restriction of f to E is measurable.

THEOREM:-

Let f be a function with measurable domain D. Then f is measurable iff the function $g(x)$ i.e.

$$g(x) = \begin{cases} f(x) & , x \in D \\ 0 & , x \notin D \end{cases}$$

is measurable.

PROOF:-

Suppose f is a measurable function. To prove g is a measurable function.

If $x \in D$ then $g(x) = f(x)$ and $f(x)$ is measurable so $g(x)$ is also measurable

If $x \notin D$ then $g(x) = 0 \Rightarrow g(x)$ is constant function and constant function is

measurable

So $\Rightarrow g(x)$ is measurable.

Conversely, suppose g is measurable
To prove f is measurable.

As g is measurable on $D \cup D'$
and $D \subset D \cup D'$
$\Rightarrow g$ is measurable on D - because g is restriction to D.
And when $x \in D$ then $f(x) = g(x)$

$\Rightarrow f$ is measurable function.

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THEOREM:- Let f be an extended real valued function with measurable domain D and let $D_1 = \{x \in D : f(x) = \infty\}$, $D_2 = \{x \in D : f(x) = -\infty\}$ then f is measurable iff D_1 and D_2 are measurable and the restriction of f to $D \setminus (D_1 \cup D_2)$ is measurable function.

PROOF:-

Let us assume f is measurable, then for all $\alpha \in \mathbb{R}$, $\{x \in D : f(x) \gtreqless \alpha\}$ is measurable.

Now, $D_1 = \{x \in D : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in D : f(x) > n\}$

and

$D_2 = \{x \in D : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in D : f(x) < -n\}$

$\Rightarrow$ D_1 and D_2 are measurable

Now $D_1 \cup D_2$ is measurable

$\Rightarrow D \setminus (D_1 \cup D_2)$ is measurable

$\Rightarrow D \setminus (D_1 \cup D_2)$ is measurable subset of D

then restriction of f to $D \setminus (D_1 \cup D_2)$ is measurable.

Conversely, assume D_1 and D_2 are measurable and restriction of f to $D \setminus (D_1 \cup D_2)$ is measurable

To prove f is measurable on D.

Let $\alpha \in \mathbb{R}$, then

$\{x \in D : f(x) > \alpha\} = \{x \in D : f(x) = \infty\} \cup \{x \in D \setminus (D_1 \cup D_2) : f(x) > \alpha\}$

As R.H.S is measurable

So L.H.S is measurable

$\Rightarrow f$ is measurable function on domain D.

THEOREM:

f and g are extended real valued measurable function and α be any fixed number then $f+g$ is measurable provided we define $f+g$ to be α whenever it is of the form $\infty - \infty$ or $-\infty + \infty$

PROOF:-

Let us define

$D_1 = \{x \in D : f(x) = \infty\} \cap \{x \in D : g(x) = -\infty\}$

$D_2 = \{x \in D : f(x) = -\infty\} \cap \{x \in D : g(x) = +\infty\}$

Then D_1 is measurable being the intersection of two measurable sets.
Similarly D_2 is measurable.

Further as $f+g$ is constant function on $D_1 \cup D_2$ with value α. Therefore $f+g$ is measurable on $D_1 \cup D_2$. Further $f+g$ is also measurable on $D \setminus (D_1 \cup D_2)$

by the fact that $f+g$ is a real valued function on $D \setminus (D_1 \cup D_2)$ ∴

Now $f+g$ is measurable function on D by the fact that $f+g$ is measurable on $D \setminus (D_1 \cup D_2)$ and $f+g$ is measurable on $D_1 \cup D_2$ and union of two measurable sets is measurable.

THEOREM:-

Any extended real valued function define on a set of measure zero is measurable.

PROOF:-

let f be defined on D, then by given condition $m^*(D) = 0$

$\Rightarrow D$ is measurable.

Now, For any real number α

$\{x \in D : f(x) > \alpha\} \subseteq D$

$\Rightarrow m^*(\{x \in D : f(x) > \alpha\}) \leq m^*(D)$

$\because m^*$ is montone

$\Rightarrow m^*(\{x \in D : f(x) > \alpha\}) \leq 0$

$\Rightarrow m^*(\{x \in D : f(x) > \alpha\}) = 0$

$\Rightarrow \{x \in D : f(x) > \alpha\}$ is measurable

$\Rightarrow f$ is measurable.

DEFINITIONS:-

(i) A property is said to hold almost everywhere (written as a.e.) if the set of points where it does not hold has measure zero.

(ii) Two functions f and g with same domain D are equal almost everywhere if measure of the set of points where they are not equal has measure zero.

Obviously $f = g$ a.e. $\Rightarrow g = f$ a.e.

(iii) A sequence $\{f_n\}$ of functions defined on E is said to converge almost everywhere to a function f if, the set of points where $\{f_n\}$ does not converge to f has measure zero.

EXAMPLES:-

Exp 1:-

Define $f: \mathbb{R} \to \{1, 2\}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}' \\ 2 & \text{if } x \in \mathbb{Q} \end{cases}$$

then $f(x) = 1$ a.e. $\because m^*(\mathbb{Q}) = 0$

Exp 2:-

Define $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}' \\ x & \text{if } x \in \mathbb{Q} \end{cases}$$

Then $f = g$ a.e. if $m^*(\mathbb{Q}) = 0$ and $f(x) \neq g(x)$ when $x \in \mathbb{Q}$

THEOREM:- let f and g be extended real valued measurable function which are finite almost everywhere then $f+g$ is measurable.

PROOF:-

Let $D_1 = \{x \in D : f(x) = \infty\} \cap \{x \in D : g(x) = -\infty\}$

$D_2 = \{x \in D : g(x) = \infty\} \cap \{x \in D : f(x) = -\infty\}$

then as f and g are finite a.e. so $m^*(D_1) = 0$ and $m^*(D_2) = 0$ then

$m^*(D_1 \cup D_2) = 0$

Now as an extended real valued function defined on a set of measure zero is measurable so $f+g$ is measurable on $D_1 \cup D_2$.

Then also $f+g$ is measurable on $D \setminus (D_1 \cup D_2)$

$\Rightarrow f+g$ is measurable on D.

THEOREM:

Let f be a measurable function with $f = g$ a.e. then g is also measurable function.

PROOF:

Put $E = \{x \in D : f(x) \neq g(x)\}$ then $m^*(E) = 0$

$\Rightarrow E$ is measurable.

$\Rightarrow E' = D \setminus E$ is measurable.

Now, obviously for any $\alpha \in \mathbb{R}$

$$\{x \in D : g(x) > \alpha\} = \{x \in D \setminus E : g(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\}$$

$$\Rightarrow \{x \in D : g(x) > \alpha\} = \{x \in D \setminus E : f(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\} \longrightarrow *$$

$(\because$ for $x \in D \setminus E \quad f(x) = g(x))$

Now as f is measurable on D and $D \setminus E = E'$ is measurable subset of D so restriction of f on $D \setminus E$ is measurable.

$\Rightarrow \{x \in D \setminus E : f(x) > \alpha\}$ is measurable

Further as a function defined on a set of measure zero is measurable so $\{x \in E : g(x) > \alpha\}$ is measurable.

$\Rightarrow$ R.H.S of $*$ is measurable being the union of two measurable sets.

$\Rightarrow \{x \in D : g(x) > 0\}$ is measurable

$\Rightarrow g$ is measurable.

LIMIT SUPERIOR AND LIMIT INFERIOR

Let $\{x_i\}$ be a sequence of real numbers and let

$$a_1 = \sup\{x_1, x_2, x_3, \dots\} = \sup_{i \geq 1}\{x_i\}$$

and

$$a_2 = \sup\{x_2, x_3, x_4, \dots\} = \sup_{i \geq 2}\{x_i\}$$

$$a_3 = \sup\{x_3, x_4, x_5, \dots\} = \sup_{i \geq 3}\{x_i\}$$

$$\vdots$$

Then $a_1 \geq a_2 \geq a_3 \geq \dots\dots$

and Let

$$b_1 = \inf_{i \geq 1}\{x_i\}$$

$$b_2 = \inf_{i \geq 2}\{x_i\}$$

$$b_3 = \inf_{i \geq 3}\{x_i\}$$

$$\vdots$$

Then Limit superior of $\{x_i\}$ is denoted and defined by

$$\overline{\lim}\{x_i\} = \inf_k a_k = \inf_k \sup_{i \geq k}\{x_i\}$$

and Limit inferior of $\{x_i\}$ is denoted and defined by

$$\underline{\lim}\{x_i\} = \sup_k b_k = \sup_k \inf_{i \geq k}\{x_i\}$$

REMARK:-

Limit of a sequence $\{x_i\}$ exists if $\overline{\lim}\{x_i\} = \underline{\lim}\{x_i\}$, then we write it $\lim\{x_i\}$.

THEOREM:- let $\{f_n\}$ be a sequence of extended real valued measurable functions with same domain D then

(i) $\overset{n}{\underset{i=1}{\operatorname{Max}}} f_i$ is measurable, for each n.

(ii) $\overset{n}{\underset{i=1}{\operatorname{Min}}} f_i$ is measurable, for each n.

(iii) $\inf_{n \in N} f_n$ is measurable.

(iv) $\sup_{n \in N} f_n$ is measurable.

(v) $\overline{\lim} f_n$ is measurable

(vi) $\underline{\lim} f_n$ is measurable

(vii) If $\lim_{n \to \infty} f_n(x) = f(x)$ then f is measurable.

PROOF:-

Put $h = \underset{i=1}{\overset{n}{Max}} f_i$, then for any $\alpha \in \mathbb{R}$

we prove

$$\{x \in D : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x \in D : f_i(x) > \alpha\}$$

Let $y \in \{x \in D : h(x) > \alpha\} \Rightarrow h(y) > \alpha$

Now $h(x) = \underset{i=1}{\overset{n}{Max}} f_i(x)$

Therefore, then there exist a j such that

$h(x) = f_j(x) \Rightarrow h(y) = f_j(y)$

Now $h(y) > \alpha \Rightarrow f_j(y) > \alpha$

$\Rightarrow y \in \{x : f_i(x) > \alpha\}$, for some i.

$\Rightarrow y \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$

$\Rightarrow \{x \in D : h(x) > \alpha\} \subseteq \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\} \longrightarrow (1)$

Let $y \in \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$ for some i

$\Rightarrow f_i(y) > \alpha$

Now as $h(x) = \underset{i=1}{\overset{n}{Max}} f_i(x)$

$\Rightarrow h(x) \geq f_i(x)$, for each i

$\Rightarrow h(x) \geq f_i(x)$

$\Rightarrow h(y) \geq f_i(y) > \alpha$

$\Rightarrow h(y) > \alpha$

$\Rightarrow y \in \{x \in D : h(x) > \alpha\}$

$\Rightarrow \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\} \subseteq \{x : h(x) > \alpha\} \longrightarrow$ (ii)

from (i) and (ii)

$\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}$

Now as f_i is measurable and union of finite measurable sets is measurable so R.H.S is measurable
Hence, L.H.S is measurable
$\Rightarrow h$ is measurable.
$\Rightarrow \operatorname{Max}_{i=1}^{n} f_i$ is measurable.

(ii) $\operatorname{Min}_{i=1}^{n} f_i$ is measurable for each n.

Put $g = \operatorname{Min}_{i=1}^{n} f_i$

Then we prove that
$\{x : g(x) > \alpha\} = \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$

let $y \in \{x : g(x) > \alpha\} \Rightarrow g(y) > \alpha$

Now as $g(x) = \operatorname{Min}_{i=1}^{n} f_i(x)$, then there exist at least one i such that

$g(x) = f_i(x)$

and $g(x) \leq f_i(x)$, $\forall\ i$

$\Rightarrow g(y) \leq f_i(y)$

$$\Rightarrow f_i(y) \geq g(y) > \alpha$$

$$\Rightarrow f_i(y) > \alpha$$

$$\Rightarrow y \in \{x : f_i(x) > \alpha\}$$

$$\Rightarrow y \in \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

$$\Rightarrow \{x : g(x) > \alpha\} \subseteq \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\} \longrightarrow \text{(i)}$$

let $y \in \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$

$$\Rightarrow y \in \{x : f_i(x) > \alpha\}, \ \forall \ i$$

$$\Rightarrow f_i(y) > \alpha \quad \text{for each } i$$

As $g(x) = \operatorname{Min}_{i=1}^{n} f_i(x)$

Then there exist some j such that

$$g(x) = f_j(x)$$

$$\Rightarrow g(y) = f_j(y) > \alpha \Rightarrow g(y) > \alpha$$

$$\Rightarrow y \in \{x : g(x) > \alpha\}$$

$$\Rightarrow \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\} \subseteq \{x : g(x) > \alpha\} \longrightarrow \text{(ii)}$$

From (i) and (ii)

$$\{x : g(x) > \alpha\} = \bigcap_{i=1}^{n} \{x : f_i(x) > \alpha\}$$

As R.H.S is measurable.

Therefore L.H.S is also measurable

$\Rightarrow g$ is measurable

$\Rightarrow \underset{i=1}{\overset{n}{\text{Min}}}\, f_i$ is measurable.

(iii) $\underset{n\in N}{\text{Inf}}\, f_n$ is measurable.

Put $g = \underset{n\in N}{\text{Inf}}\, f_n$

Now we prove that

$\{x : g(x) > \alpha\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$

let $y \in \{x : g(x) > \alpha\}$. $\Rightarrow g(y) > \alpha$

Now as $g = \underset{n\in N}{\text{Inf}}\, f_n$

$\Rightarrow g(y) \leq f_n(y) \quad , \forall n$

$\Rightarrow f_n(y) \geq g(y) > \alpha$

$\Rightarrow f_n(y) > \alpha$

$\Rightarrow y \in \{x : f_n(x) > \alpha\} \quad , \forall n$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$

$\Rightarrow \{x : g(x) > \alpha\} \subseteq \bigcap_{n=1}^{\infty} \{x : f_n(x) > \alpha\} \longrightarrow (i)$

let $y \in \bigcap_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$.

$\Rightarrow y \in \{x : f_n(x) > \alpha\} \quad , \forall n$

$\Rightarrow f_n(y) > \alpha \quad , \forall n$

$\Rightarrow \alpha$ is a lower bound of $\{f_1(y), f_2(y), f_3(y), \ldots\ldots\}$

But $g(y)$ is the greatest lower bound.

$\therefore \quad \alpha < g(y) \quad \Rightarrow g(y) > \alpha.$

$\Rightarrow y \in \{x : g(x) > \alpha\}$

$\Rightarrow \bigcap_{n=1}^{\infty}\{x : f_n(x) > \alpha\} \subseteq \{x : g(x) > \alpha\} \longrightarrow$ (ii)

From (i) and (ii)

$\{x : g(x) > \alpha\} = \bigcap_{n=1}^{\infty}\{x : f_n(x) > \alpha\}.$

Now as each f_n is measurable and countable intersection of measurable sets is measurable

$\Rightarrow$ R.H.S is measurable

$\Rightarrow$ L.H.S is measurable

$\Rightarrow g$ is measurable

$\Rightarrow \inf_{n \in N} f_n$ is measurable.

(iv) $\sup_{n \in N} f_n$ is measurable.

Put $h = \sup_{n \in N} f_n$

we prove that

$\{x : h(x) > \alpha\} = \bigcup_{n=1}^{\infty}\{x : f_n(x) > \alpha\}$

let $y \in \{x : h(x) > \alpha\} \Rightarrow h(y) > \alpha$

Now as $h(x) = \sup_{n \in N} f_n(x)$

$\Rightarrow h(x) \geq f_n(x) \quad , \forall n \in N$

OR $h(y) \geq f_n(y) \quad , \forall n \in N$

Now as $h(y) \geq f_n(y)$ and $h(y) > \alpha$

then there exist $m \in N$, such that

$f_m(y) > \alpha$

then $y \in \bigcup_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$

$\Rightarrow \{x : h(x) > \alpha\} \subseteq \bigcup_{n=1}^{\infty} \{x : f_n(x) > \alpha\} \longrightarrow$ (i)

let $y \in \bigcup_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$

$\Rightarrow y \in \{x : f_n(x) > \alpha\}$, for some n

$\Rightarrow f_n(y) > \alpha$

Now as $h = \sup_{n \in N} f_n$

$\Rightarrow h(y) \geq f_n(y) > \alpha$

$\Rightarrow h(y) > a \Rightarrow y \in \{x : h(x) > \alpha\}$

$\Rightarrow \bigcup_{n=1}^{\infty} \{x : f_n(x) > \alpha\} \subseteq \{x : h(x) > \alpha\} \longrightarrow$ (ii)

from (i) and (ii)

$\{x : h(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > \alpha\}$

As R.H.S is measurable so, L.H.S is also measurable

$\Rightarrow f_n$ is measurable

$\Rightarrow \sup_{n \in N} f_n$ is measurable.

(v) $\overline{\lim} f_n$ is measurable.

By definition of limit superior

$$\overline{\lim} f_n = \inf_n \left(\sup_{i \geq n} f_i \right)$$

suppose

$$F_n = \sup_{i \geq n} f_i = \sup\{f_n, f_{n+1}, f_{n+2}, \ldots\}$$

Since each f_n is measurable & we know that if $\{f_n\}$ is measurable then $\sup f_n$ is measurable, therefore each F_n is measurable, then we get $\{F_n\}$ a sequence of measurable functions. Now as $\{F_n\}$ is a sequence of measurable functions. So $\inf\{F_n\}$ is measurable $\Rightarrow \inf \sup_{i \geq n} f_i$ is measurable

$\Rightarrow \overline{\lim} f_n$ is measurable.

(vi) $\underline{\lim} f_n$ is measurable.

By definition $\underline{\lim} f_n = \sup_n \left(\inf_{i \geq n} f_i \right)$

Let

$$F_n = \inf\{f_n, f_{n+1}, f_{n+2}, \ldots\}.$$

Then $\{F_n\}$ is a sequence of measurable functions, so $\sup_n \{F_n\}$ is measurable.

$\Rightarrow \sup_n \left(\inf_{i \geq n} f_i \right) = \underline{\lim} f_n$ is measurable.

(vii) Given $\lim_{n\to\infty} f_n(x) = f(x)$ exists, then to prove f is m'able.

Since $\lim_{n\to\infty} f_n(x) = f(x)$ so,

$\overline{\lim} f_n = \underline{\lim} f_n = f$.

$\Rightarrow$ f is measurable because $\overline{\lim} f_n$ & $\underline{\lim} f_n$ both are measurable.

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SORRY

THEOREM:

let f be a measurable function and G be an open set then $\{x : f(x) \in G\}$ is measurable.

PROOF:

Given that G is an open set. Then G can be expressed as a countable union of pairwise disjoint open intervals because every non empty open set in $\mathbb{R}$ is the union of a countable collection of disjoint open intervals.

Therefore, $G = \bigcup_{n=1}^{\infty} I_n$, where $I_n =]a_n, b_n[$ are pairwise disjoint open intervals. Now as each open interval is measurable and countable union of measurable sets is measurable.

$\Rightarrow G$ is measurable being the countable union of measurable sets.

Now, as

$$\{x : f(x) \in G\} = \bigcup_{n=1}^{\infty} \left[\{x : f(x) > a_n\} \cap \{x : f(x) < b_n\}\right]$$

As R.H.S is measurable

$\Rightarrow \{x : f(x) \in G\}$ is measurable.

THEOREM:

2014 2016

Let f and g be two measurable function defined on same domain D then the following sets are measurable.

(i) $\{x : f(x) < g(x)\}$

(ii) $\{x : f(x) > g(x)\}$

(iii) $\{x : f(x) \leq g(x)\}$

(iv) $\{x : f(x) \geq g(x)\}$

(v) $\{x : f(x) = g(x)\}$

PROOF:-

(i) let $f(x) < g(x)$

As we know that between two real numbers, there exists a rational number

Therefore $f(x) < r < g(x)$

As $\{x : f(x) < g(x)\} = \bigcup_{r \in Q} \left[\{x : f(x) < r\} \cap \{x : g(x) > r\}\right]$

As R.H.S is measurable.

Therefore L.H.S is measurable

$\Rightarrow \{x : f(x) < g(x)\}$ is measurable.

(ii) let $f(x) > g(x)$

Then by rational density theorem there exist a rational number 'r' such that

$$f(x) > r > g(x)$$

As $\{x: f(x) > g(x)\} = \bigcup_{r\in\mathbb{Q}} \left[\{x: f(x) > r\} \cap \{x: g(x) < r\}\right]$

As R.H.S is measurable

$\Rightarrow$ L.H.S is also measurable.

$\Rightarrow \{x: f(x) > g(x)\}$ is measurable.

(iii) $\{x: f(x) \leq g(x)\}$

As $\{x: f(x) \leq g(x)\} = D \setminus \{x: f(x) > g(x)\}$

As D and $\{x: f(x) > g(x)\}$ are measurable and the difference of two measurable sets is measurable.

Therefore, R.H.S is measurable

$\Rightarrow$ L.H.S is also measurable

$\Rightarrow \{x: f(x) \leq g(x)\}$ is measurable.

(iv) $\{x: f(x) \geq g(x)\}$

As $\{x: f(x) \geq g(x)\} = D \setminus \{x: f(x) < g(x)\}$

As D and $\{x: f(x) < g(x)\}$ are measurable

$\therefore$ As R.H.S is measurable

$\Rightarrow$ L.H.S is also measurable

$\Rightarrow \{x: f(x) \geq g(n)\}$ is measurable.

(iv) $\{x : f(x) = g(x)\}$

As
$$\{x : f(x) = g(x)\} = \{x : f(x) \le g(x)\} \cap \{x : f(x) \ge g(x)\}$$

As R.H.S is measurable
so L.H.S is also measurable

$\Rightarrow \{x : f(x) = g(x)\}$ is measurable.

THEOREM:-

2016
2013

Let f be any real valued function defined on a measurable domain D and G is an open set in $\mathbb{R}$, then f is measurable iff $f^{-1}(G)$ is measurable.

PROOF:-

Suppose f is measurable. To prove $f^{-1}(G)$ is measurable.

As G is an open set so there exist a sequence $\{I_n\}$ with pairwise disjoint open intervals such that $G = \bigcup_{n=1}^{\infty} I_n$
where say $I_n =]a_n, b_n[=]-\infty, b_n[\cap]a_n, \infty[$

then $f^{-1}(G) = f^{-1}\left(\bigcup_{n=1}^{\infty} I_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(I_n)$

$$= \bigcup_{n=1}^{\infty} f^{-1}\left(]-\infty, b_n[\cap]a_n, \infty[\right)$$

$$\Rightarrow f^{-1}(G) = \bigcup_{n=1}^{\infty}\left[f^{-1}(]-\infty, b_n[) \cap f^{-1}(]a_n, \infty[)\right]$$

$$= \bigcup_{n=1}^{\infty}\left[\{x : f(x) < b_n\} \cap \{x : f(x) > a_n\}\right] \rightarrow (1)$$

Since R.H.S is measurable so is L.H.S

$\Rightarrow f^{-1}(G)$ is measurable.

Conversely assume $f^{-1}(G)$ is measurable. To prove $f(G)$ is measurable.

As G is any open set so in particular if $G =]\alpha, \infty[$, then

$$f^{-1}(G) = f^{-1}(]\alpha, \infty[)$$

$$= \{x : f(x) > \alpha\}$$

As given $f^{-1}(G)$ is measurable.

$\Rightarrow \{x : f(x) > \alpha\}$ is measurable

$\Rightarrow f$ is measurable.

REMARK:

Above theorem is valid if f is an extended real valued function.

PROOF: Suppose for any open set G in $\overline{\mathbb{R}}$ $f^{-1}(G)$ is measurable. To prove f is measurable.

As G is any open set so in particular
Let $G =]\alpha, \infty]$ then

$$f^{-1}(G) = f^{-1}(]\alpha,\infty[)$$
$$= \{x : f(x) > \alpha\}$$

$\Rightarrow \{x : f(x) > \alpha\}$ is measurable
$\Rightarrow f$ is measurable.

Conversely assume f is measurable
To prove $f^{-1}(G)$ is measurable.

For any open set G in $\overline{\mathbb{R}}$
As f is measurable, so for any α

$\{x : f(x) > \alpha\} = f^{-1}(]\alpha, \infty])$
$\{x : f(x) \geq \alpha\} = f^{-1}([\alpha, \infty])$

$\{x : f(x) < \alpha\} = f^{-1}([-\infty, \alpha[)$
$\{x : f(x) \leq \alpha\} = f^{-1}([-\infty, \alpha])$

are all measurable and also

$f^{-1}(]\alpha, \beta[) = f^{-1}(]\alpha, \infty[) \cap f^{-1}(]-\infty, \beta[)$ is measurable.
So then if we define
$$\Omega = \{E \subseteq \overline{\mathbb{R}} : f^{-1}(E) \text{ is measurable}\}$$
Then Ω is a σ-algebra and if G is any open set in $\overline{\mathbb{R}}$ then G is countable union of pairwise

disjoint open intervals of the form

$]\alpha, \beta[$, $]\alpha, \infty]$, $[-\infty, \alpha[\in \Omega$ etc.

Then by above argument $G \in \Omega$

$\Rightarrow f^{-1}(G)$ is measurable.

THEOREM:-

If F is closed in $\overline{\mathbb{R}}$ then $f^{-1}(F)$ is measurable iff f is measurable.

PROOF:

Assume f is measurable.

To prove $f^{-1}(F)$ is measurable.

As F is closed so $F' = G$

$\Rightarrow f^{-1}(G)$ is measurable

$\Rightarrow (f^{-1}(G))'$ is measurable.

$\Rightarrow f^{-1}(G')$ is measurable

$\Rightarrow f^{-1}(F)$ is measurable

Conversely assume $f^{-1}(F)$ is measurable

To prove: f is measurable.

As $f^{-1}(F)$ is measurable ~~and~~

$\Rightarrow [f^{-1}(F)]'$ is measurable.

$\Rightarrow f^{-1}(F')$ is measurable.

$\Rightarrow f^{-1}(G)$ is measurable,

where G is an open set

Then by previous theorem

f is measurable.

SIMPLE FUNCTION

Let E be non-empty subset of a set X. Then the characteristic function of the set E is denoted and defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}$$

ایک دفعہ کا ذکر ہے ایک جنگل میں دو بیل رہتے تھے۔
پھر وہ دونوں مر گئے۔

THEOREM:-

Show that

i) $\chi_{A \cap B} = \chi_A . \chi_B$

ii) $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A . \chi_B$

iii) $\chi_{A \cup B} = \chi_A + \chi_B$, provided $A \cap B = \phi$

iv) $\chi_{A'} = 1 - \chi_A$.

PROOF:-

(i) Case-I:

If $x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

$\chi_{A \cap B}(x) = 1$

$\chi_A(x) = 1$ and $\chi_B(x) = 1$

$\Rightarrow \chi_A . \chi_B = 1 . 1 = 1$

$\Rightarrow \chi_{A\cap B} = \chi_A . \chi_B$

Case-II:

If $x \notin A\cap B$

$\Rightarrow x \notin A$ or $x \notin B$

$\chi_{A\cap B}(x) = 0$

and $\chi_A(x) = 0$ or $\chi_B(x) = 0$

$\chi_A(x) \cdot \chi_B(x) = 0 \cdot 0 = 0$

$\Rightarrow \chi_{A\cap B}(x) = \chi_A(x) \cdot \chi_B(x).$

(ii) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A . \chi_B.$

Let $x \in A\cup B \Rightarrow \chi_{A\cup B}(x) = 1$

Now as $x \in A\cup B$, Then there are three cases.

Case-I:

If $x \in A$ and $x \notin B \Rightarrow x \notin A\cap B$

$\Rightarrow \chi_A(x) = 1$; $\chi_B(x) = 0$

and $\chi_{A\cap B}(x) = 0$

$\Rightarrow \chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}$

Case-II:

If $x \notin A$ and $x \in B \Rightarrow x \notin A\cap B$

$\Rightarrow \chi_A(x) = 0$, $\chi_B(x) = 1$ and $\chi_{A\cap B}(x) = 0$

$\Rightarrow \chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$

case III :-

If $x \in A$, $x \in B$ $\Rightarrow x \in A \cap B$.

$\Rightarrow \chi_A(x) = 1$, $\chi_B(x) = 1$, $\chi_{A\cap B}(x) = 1$

and $\because \chi_{A\cup B}(x) = 1 \longrightarrow$ (i)

$\chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x) = 1+1-1 = 1 \longrightarrow$ (ii)

From (i) and (ii)

$\chi_{A\cup B}(x) = \chi_A(x) + \chi_B(x) - \chi_{A\cap B}(x)$

$\therefore$ Hence for all the cases

$\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$

(iii) $\chi_{A\cup B} = \chi_A + \chi_B$, Provided $A \cap B = \phi$

As $A \cap B = \phi \Rightarrow \chi_{A\cap B} = 0$

Also we know that

$\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$

$= \chi_A + \chi_B - 0$

$\Rightarrow \chi_{A\cup B} = \chi_A + \chi_B$

(iv) $\chi_{A'} = 1 - \chi_A$.

if $x \in A'$ $\Rightarrow \chi_{A'}(x) = 1$ — (i)

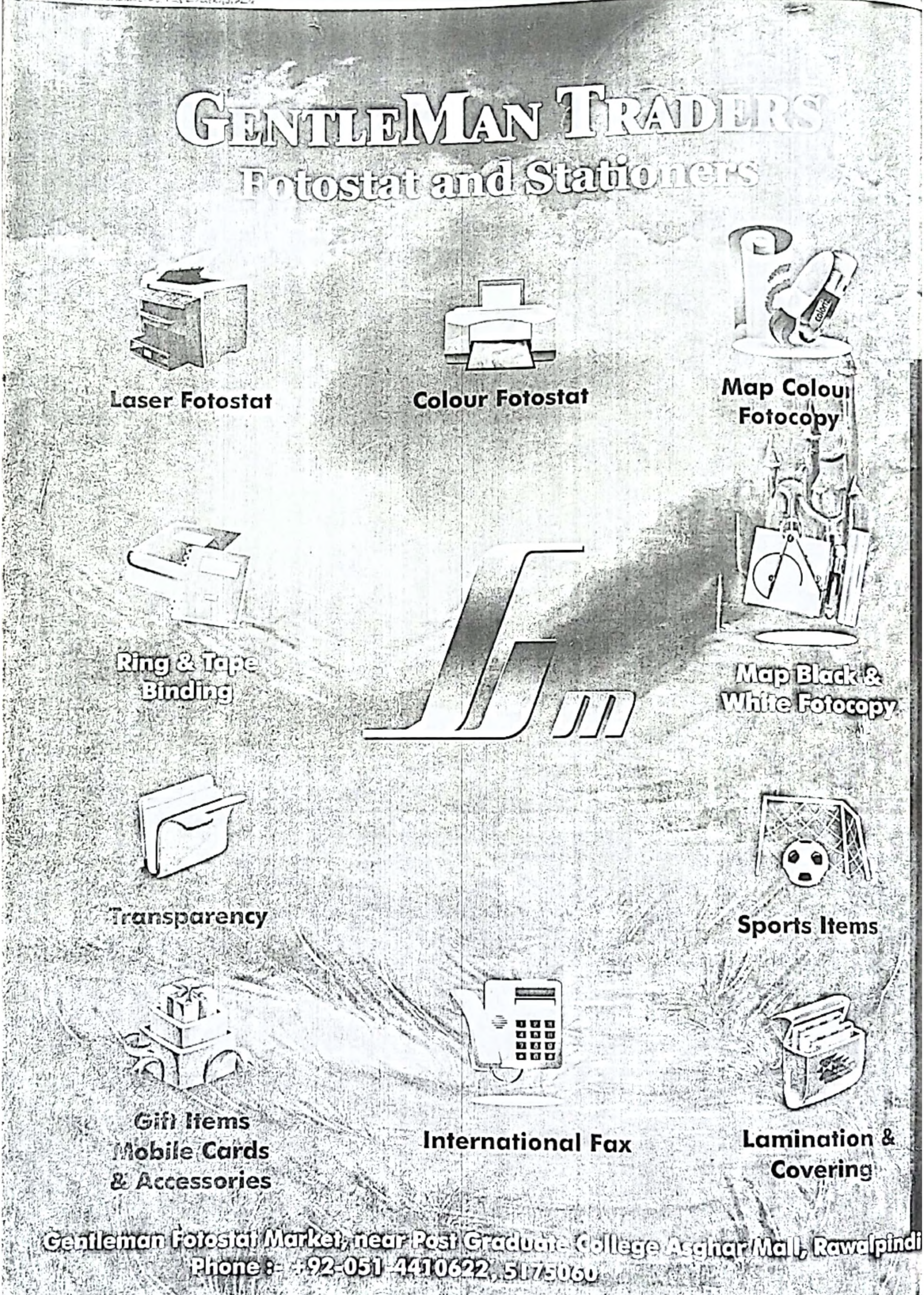
As $x \in A' \Rightarrow x \notin A \Rightarrow \chi_A(x) = 0$

$\Rightarrow 1 - \chi_A(x) = 1 - 0 = 1$ — (ii)

From (i) and (ii) we have

$\chi_{A'} = 1 - \chi_A$

GOOD LUCK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